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Teaching Modeling in Algebra and Geometry using Musical Rhythms: Teachers' Perceptions on Effectiveness

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The study was conducted within a summer institute program that trained 15 mathematics teachers from two high need school districts in facilitating students' understanding of mathematical modeling using musical rhythms. Qualitative and quantitative data was collected and triangulated using different sources. This study showed that when teachers are engaged in inter-disciplinary experiences situating mathematics in the context of music, they were confident that such experiences can be effective in stimulating student learning. Furthermore, the study argues that introducing teachers to creative ways of linking mathematical ideas with musical rhythms will not only impact their teaching outcome expectancy but will also strengthen their self-efficacy to teach mathematics in less traditional ways.

Key words: modeling in algebra, geometry, musical rhythms, and teachers' perceptions.

Mathematical modeling is one of the vital mathematics topics that all students must study in order to be college and career ready. In this respect, there is an imperative need to equip teachers with necessary skills and dispositions to transform the high school mathematics curriculum landscape in ways that facilitate students' conceptual understanding and scientific thinking in preparation for college and for the 21st century economy. Furthermore, research on juxtaposing mathematical modeling with arts, using technology-enhanced instructional strategies, has undergone a deep transformation due to opportunities offered by the rapid evolution of technology and by the development of new pedagogical approaches.

The search for ways of communicating the concepts, techniques, and language of mathematics has given rise to interdisciplinary approaches during the past decades (Fenton, 2009). The present work is situated in the context of mathematics and the arts and, in particular, in the use of musical rhythms to transmit specific mathematical knowledge. Because musical rhythms rely on notations that can be constructed using mathematical relationships, we argue that infusing the mathematics instruction with activities inspired by analyses of musical notations would provide opportunities for meaningful learning.

Despite efforts to ascertain the usefulness of integrating arts in the teaching of mathematics, the effectiveness of using music as a medium to facilitate the teaching and learning of mathematics has not been well researched. To better understand how music could affect teachers' perceptions in the effectiveness of using such contexts for teaching mathematics, we conducted a research study whose goal was to analyze the perceptions of high school mathematics teachers as a result of a teacher-training workshop devoted to the use of music as a context for mathematics education.

It is worth underlining that several of the ideas included in the materials used in the workshop have been developed and employed with the same goal by other scholars interested in this subject, and we are in debt to them (Gomez, 2012; Toussaint, 2013). The study of the perceptions of this content, in the specific context of teacher preparation, has not been carried out previously and is of special interest. The particular way in which the content was delivered, the evaluation methods used, as well as the great potential for implementation in the classroom and the suggestions we consider important for the generalization of this implementation, should be made known. We also hope that the perseverance and continuity around this particular application will eventually serve the creation of a generalized curriculum in which interdisciplinarity in the classroom, in particular that of mathematics and music, becomes a common place. Specifically, the study addresses the following questions:

1. How do teachers perceive the utility and benefits of teaching certain mathematical knowledge and techniques by using the interdisciplinary approach, in this case through music?
2. How comfortable with these innovative pedagogical techniques do teachers feel?

Overall, we chose to study perceived effectiveness of the training workshop because of (1) the difficulties of measuring learning directly, (2) perceptions may reflect more rigorously participants' discernment of making meaning of the experience, and (3) perceived learning will contribute to our knowledge of learning effectiveness.

This paper is structured as follows. The initial sections describe the context of the study, including the content of the training workshop and the participants' demographics to better understand how teachers' perceptions evolved. This is followed by the research methodology, which includes a description of the survey instrument employed. The results of the survey are then discussed. Finally, conclusions are drawn and opportunities for further research are presented.

The Study Context

The study was conducted within a summer institute program that trained 15 mathematics teachers from two high need school districts in facilitating students' understanding of mathematical modeling using musical rhythms. The objectives of the program were to engage teachers in mathematical modeling activities that would: (1) develop students' competencies to work successfully with mathematical modeling of musical rhythms; (2) motivate students to study mathematics by showing them the real-world applicability of mathematical ideas; (3) provide students with opportunities to integrate mathematics with music; and (4) increase the scope of the mathematical content and the range of problem situations that involve computing, constructing, modeling, and visually representing musical rhythms.

The broad purpose of the study was to evaluate the effectiveness of the use of music as a context for mathematics education. However, in this article we will focus on teachers' perceptions of the effectiveness of the workshop. Our motivation comes from the fact that if teachers are not convinced of the effectiveness of the new techniques, or if they do not feel comfortable with their level of expertise, the success and implementation of the material elaborated for the classroom will be compromised.

The ultimate goal of the summer institute was to equip teachers with the necessary knowledge and dispositions to facilitate students' problem solving and modeling skills and to enhance their motivation toward achieving higher-order learning outcomes. The 36-hour professional training focused on research-based instructional strategies for teaching applied mathematical modeling to help high school students develop problem solving skills using multidisciplinary contexts. Eighteen of these hours (two complete days and one half day) were devoted to topics using the interrelationships between mathematics and music at the pedagogical level. The theme of this part of the workshop was presented as *Modeling Musical Rhythms using Algebraic and Geometric Representations*. The mathematics topics included: greatest common divisor and Euclid's Algorithm, timing systems in neutron accelerators, Euclidean rhythms, maximal evenness, rhythm and rotations, geometrical analysis of cyclic rhythms and clapping music and permutations.

During training, teachers were engaged in guided discussions and reflections that focused on developing lesson plans to foster their conceptual understanding. The idea was to build interrelationships between mathematics and music so as to describe and explain relevant concepts and skills proposed in the high school mathematics curriculum.

The content was delivered by the second author, a researcher in Mathematical Music Theory at a public research university in the southeastern United States. A student from the School of Music served as an assistant and was instructed to illustrate the mathematical concepts using rhythms. During hands on explorations of the activities, the music student worked with the teachers individually in the group sessions where the rhythms were practiced with simple percussion instruments and clapping. Group sessions were intertwined with the lectures so that the information could be assimilated and worked upon, both theoretically and practically through the musical illustrations. The coordinator of the workshop, the first author, a mathematics educator and specialist in ethnomathematics was present during all sessions and participated in the group activity. At the end of each day, “wrap-up” sessions were conducted and written questionnaires, which have been quantified, were applied to the participants. The analysis of these questionnaires will shape the sections devoted to results and conclusions.

During the final hours of the workshop (the end of the second day and the beginning of the last day) the teachers had to develop a project in groups of two, to be presented at the end of the final session. They were asked to incorporate into a mathematics lesson some of the ideas and techniques seen in the workshop, using a relevant algebra or geometry topic and to justify the inclusion of the material learned in the workshop as a form of modeling.

Content of the Training

The purpose of this section is to give the reader synthesized idea of the content of the different sections of the workshop. By no means do we wish to enter into great detail, given that the majority of the topics and their development can be found in the references that we provide. However, the main idea of this article is the communication of the results of the workshop through the analysis of the perception, participation, and contribution of the teachers. We also want to analyze the perspectives of music in the mathematics classroom that arise from this experience, and it is essential that the reader has an overall idea of the content when arriving to those comments.

Greatest Common Divisor and Euclidean Rhythms

Detailed descriptions of the theoretical development underlying the subject of this unit, as well as other pedagogical experiments, can be found in Demaine, Gómez, Meijer, Rappaport, Taslakian, Toussaint, Winograd, and Wood (2009), Gomez (2012), and Toussaint (2013). The main mathematical concepts are as follows:

When an integer a is divided by an integer b the result is a quotient q and a remainder r which can be zero. It is shown that the greatest common divisor of a and b is the same as the greatest common divisor of q and r. Using this fact, the Euclidian algorithm distributes the division until finally arriving to a number r_{k+1} that evenly divides r_k; such a number is the *greatest common divisor* of a and b.

It was noted (Demaine et al., 2009) that Björklund’s algorithm, used in the timing systems of Spallation Neutron Source Accelerators, was equivalent to Euclid’s algorithm, as can be observed in Figure 1. It deserves mentioning that the most interesting distributions will occur when a and b are relatively prime and the distribution is not even, but *maximally even.*

The concept of *maximal evenness* arose in the context of Mathematical Music Theory and was later applied in Physics. It is an interesting and not very well known fact that this theoretical discovery from Music Theory spurred on a connection to the one dimensional antiferromagnetic $\frac{1}{2}$ Ising spin model, resulting in alternative calculations published by Krantz, Douthett, and Doty (1998) and Douthett and Krantz (2007) in the *Journal of Mathematical Physics*. The original motivation arose from the question "why are the black and white notes of the piano distributed as they are?" (Johnson, 2008). The seven white notes, representing the usual seven note diatonic scale and the seven modes, are distributed in a maximally even way within the twelve note chromatic scale; the five black notes, representing the pentatonic scale and its five modes, are also distributed in a maximally even configuration. Therefore, by using the Euclidian and/or Björklund algorithm, the question about the distribution of the notes has a logical mathematical explanation.

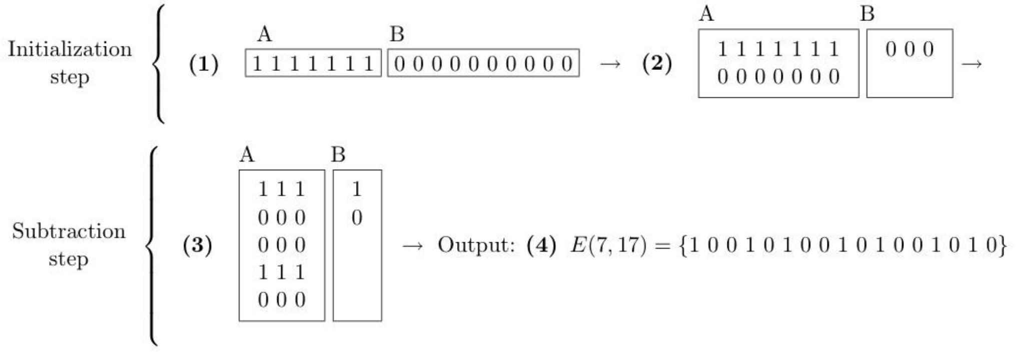

Figure 1. **Euclid's and Björklund´s algorithm to generate a maximally even rhythm with 7 "notes" (strong beats) and 17 pulses.**

These concepts were introduced, studied, and practiced by the teachers in the workshop. Moreover, we incorporated in the training the most popular world rhythms that have a maximally even distribution of "notes" (strong beats) within a timeline of pulses. These rhythms are known as Euclidian rhythms. In Figure 2, a maximally even distribution of seven strong beats in a timeline of twelve pulses is shown. In fact, this is an African Euclidian rhythm whose name is Bembé together with its rotation known as Bembé-2.

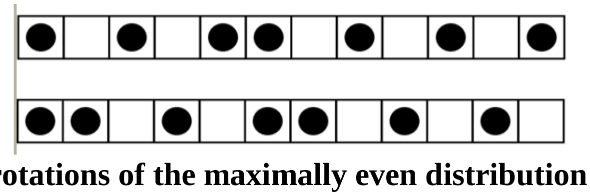

Figure 2. **Two rotations of the maximally even distribution of seven strong beats in twelve pulses.**

Geometrical Analysis of Cyclic Rhythms

Once the timeline is defined, the rhythm can always be represented by points on the circumference of a circle. Thus, in Figure 3, we can present the bembé rhythm along with several rotations that are also classified in musicology.

In Figure 3, we see the twelve points evenly distributed around the circles and, in the center, a heptagon formed by connecting the points that correspond to the distances as in the first rhythm of Figure 1, the bembé rhythm. Then, taking the rotations clockwise, we get the other rhythms. That is, the bembá is a 60 degree rotation of the bembé, the tambú is a 150 degree rotation, the Yoruba a 210 degree rotation and the bembé-2, which is the second rhythm of Figure 1, is a 330 degree clockwise rotation. Of course, we can take the rotations counterclockwise as well and end up with the same rhythms. From the geometry of rotations perspective, it is natural to continue with cyclic permutations, which leads to the subject of the next lesson.

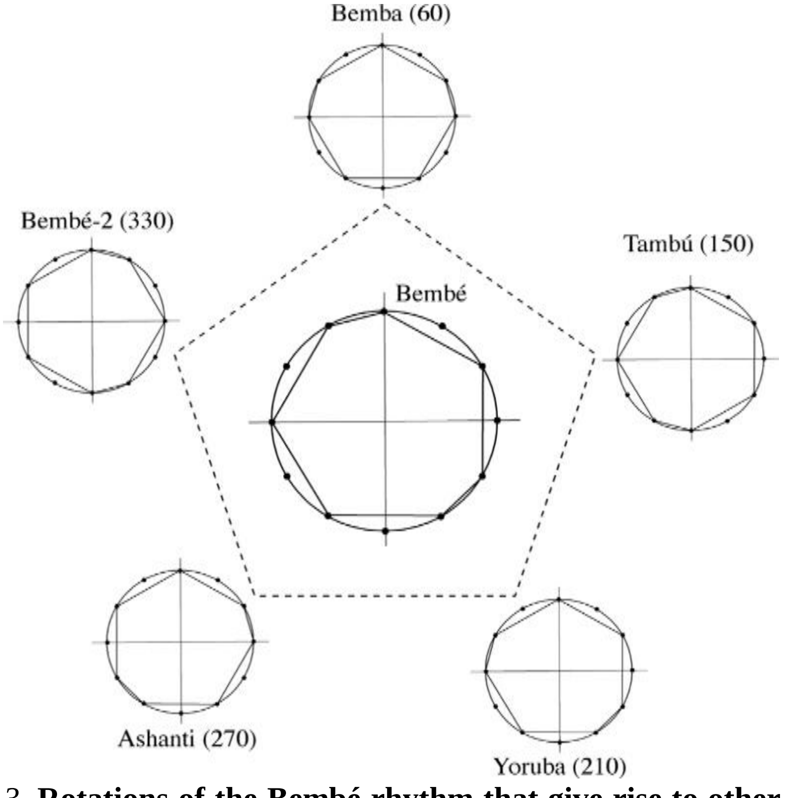

Figure 3. **Rotations of the Bembé rhythm that give rise to other classified rhythms.**

Clapping Music and Cyclic Permutations

Clapping Music by composer Steve Reich is a change of phase piece for two performers that only clap hands. Each one of the performers plays the same pattern throughout the piece and the change of phase is discrete, with one performer that begins the pattern from a different point in time, and advances after a few repetitions of the pattern while the other performer remains imperturbable, playing the same pattern without any change. There are antecedents of Clapping Music in the classroom (Haack, 1991), the majority at the university level, given that there are aspects of group theory and combinatorics implicit in the selection of the pattern that was chosen.

In the workshop, we also studied the selection of the particular pattern over other possibilities; however, we situated these concepts in the context of the content standards that the teachers are teaching in their classes at the high school level. Teachers saw that, as the pattern rotated and the phases changed, a series of intertwined rhythms were created with a wide range of rhythmic variety. We also pointed out certain symmetries created by several variations and emphasized the fact that once the pattern is defined the rules are very rigid, hence not any pattern would create the same aesthetic effect.

In Figure 4, we see the Clapping Music score. As an example of the focus of the lesson, we looked at some of the measures, where each measure represents a variation. For example, the variations 0 and 12 (V_0 and V_{12}) are trivially equal, as the cycle can only go through twelve phases at the most and, due to the wise selection of the pattern, the twelve phases are different in Clapping Music. On the other hand, variation one is equivalent to variation 11 ($V_1 \sim V_{11}$), but they are not equal. However, after hearing them several times we perceive them as equal. This is due to the fact that V_{11} is essentially the same as V_1 with the performers switched. There are also other interesting interchanges and symmetries that arise when the phases caused by the permutations reach the different measures.

Figure 4. **Clapping music score.**

At the conclusion of the summer institute, participating teachers returned to their schools with the intention to implement their learning experiences with musical activities in their respective mathematics classrooms. The consistency of the training and the materials provided to each school created an optimal situation to ascertain the effectiveness of using music in teaching mathematics as perceived by the teachers.

Methodology

In this study, qualitative and quantitative data was collected and triangulated using different sources. In order to study training effectiveness of using music as a context for teaching, we employed Mathematics Teaching Efficacy Belief questionnaire (Enochs et al., 2000). Perceived effectiveness of training was measured using two scales: *Personal Teaching Efficacy* scale and *Teaching Outcome Expectancy* scale. We argue that these two scales signify teacher perceptions regarding the effectiveness of using music in teaching mathematics (See Figure 5).

Participants

The sample comprised of 15 high school mathematics teachers, 9 females and 6 males with varied years of teaching experience ranging from 6 to 30 years. Four of the teachers were completing their graduate degrees; one teacher had a PhD in mathematics. The teachers were nominated by the two high-need Local Educational Agency (LEA), which were targeted for inclusion in the study. According to the U.S. Department of Education specification, a high-need LEA is defined as per two criteria. The first criterion involves student body and it necessitates that LEA serves no fewer than 10,000 children or for which not less than 20 percent of the children served by the agency are from families with incomes below the poverty line. The second criterion involves teachers and it includes LEA for which there is a high percentage of teachers not teaching in the academic subjects or grade levels in which they were trained to teach or for which there is a high percentage of teachers with provisional or temporary certification or licensing.

Instrument

The instrument consists of 21 items, 13 items on the Personal Mathematics Teaching Efficacy (PMTE) scale and 8 items on the Mathematics Teaching Outcome Expectancy (MTOE) scale. Each item is scored on a 5-point Likert scale response ranging from 1= Strongly Disagree to 5 = Strongly Agree. The instrument was administered at the end of the workshop.

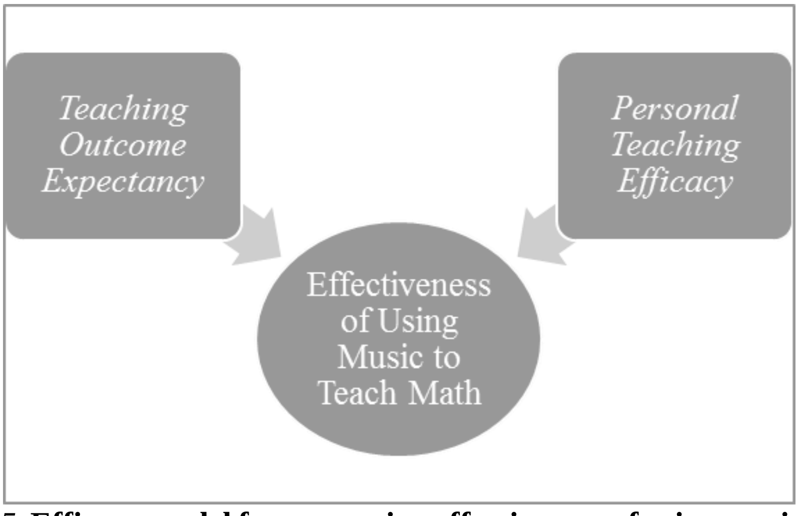

Figure 5. **Efficacy model for measuring effectiveness of using music to teach mathematics**

Qualitative data comprised teachers' narrative responses to open ended questions to elicit the situated meaning constructed and the social language used by teachers to reflect on their experiences and to represent their ideas. We were particularly interested in extracting discourse patterns in teachers' short narratives that depict the extent to which musical contexts shaped their understanding of the teaching practice.

Results

We generated descriptive statistics on each item comprising the MTEBI questionnaire in response to the study's questions. These descriptive statistics include the frequency distributions of teacher responses on the 5- point Likert scale ranging from 1= Strongly Disagree to 5= Strongly Agree with regard to 21 survey items for the two subscales, namely *Teaching Outcome Expectancy* and *Personal Teaching Expectancy* tested (See Table 1 below).

Furthermore, we examined teachers' narrative responses to the summer institute feedback survey which comprised of 10 open-ended questions asking teachers to reflect on their learning experiences as they engage in solving problems within the music context (See appendix). The first five questions required participants to provide both quantitative and qualitative feedback regarding the effectiveness of the summer institute. The second five questions asked teachers to comment on the nature of music activities chosen readings, and teaching methods used. Table 2 below summarizes teacher evaluation scores.

Table 1

Means Scores of Teachers' Responses across the Two Effectiveness Subscales

Sub-scale	**Items on MTEBI**	**Mean Score**
Personal Mathematics Teaching Efficacy (PMTE)	1. I will continually find better ways to teach mathematics	4.5
	2. Even if I try hard, I will not teach mathematics as well as I will m subjects	1.5
	3. I know how to teach mathematics concepts effectively	4.3
	4. I will not be very effective in monitoring mathematics activities.	2.2
	5. I will generally teach mathematics ineffectively	1.5
	6. I understand mathematics concepts well enough to be effective in teaching	4.5
	7. I find it difficult to use manipulatives to explain to students why math works.	2.4
	8. I will typically be able to answer students' questions	4.7
	9. I wonder if I will have the necessary skills to teach mathematics.	3.5
	10. When a student has difficulty understanding a math concept, I will usually be at a loss as to how to help the student understand it better.	2.6
	11. Given a choice, I will not invite the principal to evaluate my math teaching	2.3
	12. When teaching mathematics, I will usually welcome student questions	5
	13. I do not know what to do to turn students on to mathematics	2
Mathematics Teaching Outcome Expectancy (MTOE)	1. When a student does better than usual in mathematics, it is often because the teacher exerted a little extra effort	4 .
	2. When the mathematics grades of students improve, it is often due to their teacher having found a more effective teaching approach.	4
	3. If students are underachieving in mathematics, it is most likely due to ineffective mathematics teaching.	4
	4. The inadequacy of a student's mathematics background can be overcome by good teaching.	2.8
	5. When a low-achieving child progresses in mathematics, it is usually due to extra attention given by the teacher.	3.6
	6. The teacher is generally responsible for the achievement of students in mathematics.	4.2
	7. Students' achievement in mathematics is directly related to their teacher's effectiveness in mathematics teaching.	3.2
	8. If parents comment that their child is showing more interest in mathematics at school, it is probably due the performance of the student's teacher.	3.8
		4.2

A closer examination of the mean scores on each of the items measuring perceived teachers' self-efficacy and teaching outcome expectancy showed that teachers had positive views on the use of music as a context for effective teaching and learning. Furthermore, qualitative feedback indicated that participants appreciated the use of music activities to facilitate the teaching of mathematics concepts. As one teacher noted, "I have greater understanding of the relationship between music and mathematics via Euclidean algorithms." In particular, teachers were highly appreciative of the role that mathematics plays in modeling musical rhythms. For example, a teacher explained: "I was impressed to witness a common childhood beat receive a symbolic mathematical representation." With respect to integrating music in learning mathematics, one of the teachers expressed her interest in engaging students in using music –related situations to solve problems, she wrote: " I hope to use the music activity in my own classroom, since several of my students are musically talented, I think this will really spark their interest."

Table 2

Summary of Rating Scores for Evaluation Survey

Question	Frequency of scores (n=15 for questions 1, 2, 3,4, 5) 4 High	5 Very High	Mean Rating score
1. To what extent do you feel the goals/objectives for his workshop were accomplished?	2	13	4.87
2. To what extent do you feel the time in the workshop was sufficient to allow learning and practicing new concepts?	5	10	4.67
3. How would you rate the overall effectiveness of the instructors – preparations, style, methods, rapport – for this workshop?	2	13	4.87
4. To what extent did the workshop provide you with useful ideas which you expect to apply in your teaching?	2	13	4.87
5. What overall rating would you give to the workshop?	1	14	4.93

Overall, teachers expressed a genuine interest toward incorporating musical rhythms as contexts to engage in meaningful problem solving. They consistently referred to the idea that modeling musical rhythms using mathematics can to a large extent enhance the learning process. Seeing mathematical concepts embodied in the context of music helped teachers understand the power of mathematics in explaining real life phenomena. At the end of the summer institute, teachers were more confident in their ability to implement the lessons they created during the summer institute. However, teachers indicated that institutional support is key to ensuring meaningful and sustainable implementation of novel experiences in the classrooms, as one teacher argued, “I can definitely see the mathematical relevance of [using music in mathematics teaching] but I’m not sure how to tie it into the standards that we are required to teach.”

Conclusion

Like music, mathematics has had profound effect on the cultures that created it. The study of mathematics, like that of music, enriches our understanding of our cultural environments. Music can provide a rich, inspiring, and authentic context in which meaningful learning emerges. Many critics worldwide agree that the currently fashionable methods of accountability and standards have perpetuated a climate of compounded tension on mathematics teachers (Chahine & King, 2012). Predictably, teachers more than ever feel the need to continuously change their pedagogies to make sure that students acquire the necessary skills and content knowledge proposed by the Common Core Standards Curricula.

This study showed that when teachers are engaged in inter-disciplinary experiences situating mathematics in the context of music, they were confident that such experiences can be effective in stimulating student learning. Furthermore, the study argues that introducing teachers to creative ways of linking mathematical ideas with musical rhythms will not only impact their teaching outcome expectancy but will also strengthen their self-efficacy to teach mathematics in less traditional ways. When students are engaged in modeling musical rhythms, teachers are more likely to cultivate substantial diversity in student thinking. However, to build meaningful connections in schools between mathematical ideas and music, there must be an institutional commitment to facilitate interdisciplinary participation between the mathematics department and the music department. Without support, it would be difficult for teachers to implement new ways of teaching in their classrooms.

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