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Summing Up: Cognitive Flexibility and Mental Arithmetic

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We introduce this special issue on cognitive flexibility in mental arithmetic by first summarizing what is known. That is, we briefly examine the available literature on cognitive flexibility in terms of empirical findings and their theoretical underpinnings. Second, we describe in moderate detail elements of our research to exemplify some of the features of a relatively mature research program. Third, we provide a set of modest proposals for future research programs on cognitive flexibility in mental calculation. Fourth, we give a summary orientation for each of the four contributions that comprise the remaining body of this special issue. Heinzeet al.provide an analysis of previously published empirical research to compare the relative effects of explicit versus implicit teaching aimed at promoting cognitive flexibility. Corso et al. report on their transcultural replication with Brazilian students of certain elements of the Rathgeb-Schnierer and Green (2013, 2015, 2017a, 2017b) research originally conducted on American and German elementary students. Serrazina and Rodrigues report a case study of teacher interventions during arithmetic class that produces improvements in specific aspects of cognitive flexibility. Finally, Korten describes how student-to-student interactions, with heterogeneous pairs of students, can lead to improvements in cognitive flexibility. These contributors also map out areas of need for researchers interested in pursuing the topic.

***Keywords*:** cognitive flexibility, addition and subtraction, mental arithmetic, number patterns, problem characteristics

Interest in the relationship between cognitive flexibility and mental arithmetic has early roots in the problem-solving program initiated by Polya (1945). What Polya did was to open for examination the new and unfocused field of problem-solving, broadly defined. While logic underpinned the basic principles of mathematics, he promoted heuristics elements of reasoning using aspects of intuition, similarity, and proximity as a means for achieving solutions to problems. His pioneering work is considered groundbreaking in at least two respects. First, he showed how elements of reasoning could be systematically applied to non-arithmetic problems. Second, he extended the field of mathematics beyond classical traditions of arithmetic, geometry, algebra, and calculus. Moreover, his pedagogical advice underscores this special edition when he suggests that teachers, rather than emphasizing routine computation, should work hard to enhance students' imagination and mental strategies (Polya, 1957). Such advice presages this special edition by three-quarters of a century!

At the beginning of the twenty-first century, the National Council of Teachers of Mathematics (NCTM) argued that students should be able to use a wide variety of problem-solving strategies and that they should be able to adjust familiar strategies as well as invent new ones (NCTM, 2000). Critical to the effective use of mental strategies is cognitive flexibility, an attitude of mind that is both adaptive and agile. In this context, the past decade has seen significant gains in our understanding of mental processes that contribute to mental flexibility and adaptive expertise (e.g., Baroody, 2003; Hatano, 2003; Rathgeb-Schnierer & Green, 2013; Selter, 2009; Threlfall, 2009; Verschaffel et al., 2009). Also, different approaches have been invented for conducting empirical research on flexibility in mental arithmetic (e.g., Rathgeb-Schnierer & Green, 2013, 2015; Torbeyns et al., 2002; Torbeyns et al., 2009; Torbeyns & Verschaffel, 2016).

Theoretical Background

Research

Elements of "mental calculation" have occupied researchers for much of the second half of the twentieth century. With the beginning of the twenty-first century, researchers have examined different elements of flexibility in mental addition and subtraction and reported a variety of results:

- After learning a standard computing algorithm, students tend to prefer those, and stop using previously learned number-based strategies even when they are more advantageous and appropriate (Hickendorff, 2018; Selter, 2001; Torbeyns & Verschaffel, 2016).
- Learning strategies sequentially by example seems to have a negative impact on the development of flexibility. Students not only acquire specific procedures rather than general rules, but also tend to remain with the one strategy which was taught at first (Heirdsfield, & Cooper 2004; Klein & Beishuizen, 1998; Schütte, 2004).
- Students' strategy use depends on various factors, such as the target operation (Torbeyns et al., 2009), specific numerical or problem characteristics (Blöte et al., 2000; Torbeyns et al., 2009), and students' recognition of number patterns, problem characteristics, and relationships (Macintyre & Forrester, 2003; Peltenburg et al., 2011; Rathgeb-Schnierer, 2010; Rathgeb-Schnierer & Green, 2015, 2017a, 2017b; Threlfall, 2009).
- In the domain of solving equations, students' potential flexibility is significantly higher than their practical flexibility (Xu et al., 2017).

- Flexible, adaptive expertise in mental calculations are associated with several abilities (e.g., deep understanding of number relationships and arithmetic operations, knowledge of basic facts and fact families) and effects (e.g., high self-confidence, a positive attitude towards mathematics) (Heirdsfield & Cooper, 2002, 2004; Threlfall, 2002).
- Specific approaches to math education support the development of flexibility in mental calculation. For example, a problem-solving approach is more advantageous than an investigative approach to fostering mental flexibility (Heinze et al., 2009; Heinze et al., 2015). Moreover, an interleaved approach to teaching subtraction in third grade, combined with prompts to compare strategies, promotes students' flexible and adaptive use of subtraction strategies (Nemeth, et al., 2019). Finally, students of all achievement levels improve their cognitive flexibility in addition and subtraction when systematically engaged in *Zahlenblickschulung,* an approach to problem-solving designed to promote number sense and metacognitive competencies (Rathgeb-Schnierer, 2010; Rechtsteiner-Merz, 2013)
- Students with learning difficulties in arithmetic need special instructional approaches to develop flexibility in mental calculations (Verschaffel et al., 2007). They exhibit conceptual progress from a particular approach to math education (*Zahlenblickschulung*) that incorporates opportunities to discover, construct, organize, and evaluate numerical patterns and relationships (Rechtsteiner-Merz, 2013; Rechtsteiner-Merz & Rathgeb-Schnierer, 2015, 2017).
- German and American elementary students exhibit similar repertoires and patterns of cognitive flexibility with multi-digit addition and subtraction problems (Rathgeb-Schnierer & Green, 2015, 2017b).

The special edition presented here brings together the work of several research teams currently investigating elements of cognitive flexibility, its role in and relationship to mental arithmetic, and the implications of such work for arithmetic pedagogy. These efforts underscore both the variety of research interests and approach currently employed by researchers.

Definitions of Flexibility

Research on cognitive flexibility reflects not only different interests and aims but also different definitions of flexibility that influence both the research methods used and the manner of data interpretation. Inconsistent perspectives appear in the research literature on mental calculation flexibility (Rathgeb-Schnierer & Green, 2013; Star & Newton, 2009). For example, nearly all definitions have the same basic idea of flexibility in mental calculations as an appropriate way of acting when faced with a problem based on a repertoire of available strategies, which is to say that flexible strategies are adapted dynamically to problem situations. Nevertheless, there exist in current research crucial differences concerning the meaning of what constitutes *appropriate* as well as the use of different methods to measure *flexibility* and *appropriate ways of acting* (Rechtsteiner-Merz, 2013). Many researchers define flexibility as the choice of the most appropriate solution to a problem (Star & Newton, 2009; Verschaffel et al., 2009). While Torbeyns et al. (2009) had the same notion, they enhanced their definition to incorporate both computational accuracy and timeliness: "strategy flexibility is conceived as selecting the strategy that brings the child most quickly to an accurate answer to the problem" (Torbeyns et al., 2009, p. 583). A recent report by Rechsteiner-Merz (2013) systematically analyzed the various notions of flexibility in the literature and identified two main perspectives on what exactly is meant by the adaptive use of strategies and how this can be identified (see also Rechtsteiner & Rathgeb-Schnierer 2017). First, it is the match of solution methods and problem characteristics which becomes apparent by the conscious or unconscious choice of the most appropriate solution to a specific problem and is measured by (a) accuracy and speed (e.g., Verschaffel et al., 2009) as well as (b) the number of solution steps (e.g., Star & Newton 2009). Second, it is an emphasis and focus on cognitive elements that underlie the solution process. This means flexibility and adaptive use of strategies are identified by revealing if a learned procedure or recognized characteristics and numerical relations of a given problem sustain the solution process (e.g., Therelfall 2002, 2009; Rathgeb-Schnierer & Green, 2013, 2015).

One way to set a context for this special edition is to review important elements of our research program that address a variety of factors in the domain of cognitive flexibility and which has been presented through various publications (Rathgeb-Schnierer & Green 2013, 2015, 2017a, 2017b). We highlight some of the important elements of this work in summary-form next.

A Mature Research Program in Cognitive Flexibility

The various reports of our research rely on a concept of flexibility that views multiple elements as operating during any process of mental calculation (see Rathgeb-Schnierer, 2011; Rathgeb-Schnierer & Green, 2013). It also derives in part from current research results that underscore the critical role played by number patterns and relationships in promoting cognitive flexibility in mental arithmetic. In this context, Rathgeb-Schnierer and Green (2013) define flexibility as *cognitive actions that match the combination of strategic means to the recognized number patterns and relationships of a given problem in the context of processing a problem solution*. This definition is similar to Threlfall's (2002) "interaction between noticing and knowledge" (p. 29).

Within the past decade, we have initiated a cross-national study of cognitive flexibility in elementary students. The work proceeded in three phases. Phase 1 was dedicated to designing a new methodology for eliciting non-algorithmic cognitive strategies in the context of sorting and then solving arithmetic problems. This phase detailed the theoretical bases of our approach, and it incorporated data collection and preliminary analyses for second and fourth grade German and American students. In Phase 2, we identified the variety and frequency of flexible mental strategies, established a theoretical structure for encoding empirical data, and compared elements of cognitive flexibility between German and American second and fourth graders. This phase resulted in the identification of three profiles in students' use of cognitive flexibility when performing mental arithmetic. In Phase 3 we have begun to map out the implications of their work for mathematical pedagogy. Important components from each of these phases are briefly summarized below.

Phase 1 – Development of a New Research Methodology

We set out to establish a rigorous but robust methodology for eliciting and analyzing elementary students' cognitive flexibility exhibited in the context of sorting multi-digit addition and subtraction problems. To be clear, they aimed to examine mental processes that underlie problem-solving in terms of active cognitive elements. Their methodology was designed to assess whether students recognized problem characteristics, number patterns, and number relationships, as well as whether or not they used this knowledge in solving a problem (Rathgeb-Schnierer & Green 2013, 2015, 2017a, 2017b). In short, we sought to identify *cognitive elements* that support mental arithmetic.

Given that context, we examined directly whether students recognized problem characteristics, number patterns, and relationships, and whether they used this knowledge for solving a problem (Rathgeb-Schnierer & Green, 2013). However, our definition of flexibility differentiates between solution processes based on learned procedures (i.e., step-by-step mental calculations) versus recognized problem characteristics, number patterns, and relationships. This distinction focused methodological design elements on identifying the variety of cognitive elements that support mental arithmetic.

Problem Sorting

Arithmetic problems were needed that comprise various problem characteristics and would elicit one or more specific reasoning strategies based on recognized patterns and characteristics if they were cognitively available to a student. Reasoning strategies not available could not be elicited, no matter how suggestive the characteristics of a problem might appear. Conversely, if patterns and characteristics are recognized, and a reasoning strategy is cognitively available to a student, it is most likely to be elicited in a situation specifically constructed to evoke the strategy. The following twelve two-digit addition and subtraction problems were selected because their structure was believed likely to elicit reasoning based on problem characteristics if they were (a) recognized and (b) those reasoning strategies were cognitively available.

- 33+33: no regrouping, double digits; double facts in the ones place; the inverse of 66-33
- 34+36: regrouping; double facts in the tens place; ones add up to ten
- 47+28: regrouping
- 56+29: regrouping; 29 close to thirty
- 65+35: regrouping; fives in the ones place add up to ten
- 73+26: no regrouping
- 31-29: renaming; range of numbers; 29 is close to thirty
- 46-19: renaming; 19 is close to twenty
- 63-25: renaming
- 66-33: no renaming; double and half relation; double digits; the inverse of 33+33)
- 88-34: no renaming; double and half relation of the ones
- 95-15: no renaming; fives in the ones place

Each problem was printed on 3 x 5 cards and arrayed randomly on a desk in front of the student. Students were asked to first examine all the cards and then place each one either on the side labeled "Easy" or on the side labeled "Hard."

Flexibility-Eliciting Interviews

Semi-structured interviews were developed to elicit students' reasoning for sorting problems as "easy" or "hard." After sorting a card, students were asked, "Why is this problem easy/hard for you?" The sequence of questions depended on the order in which students conducted their free sort of the 12 available cards. Each interview consisted of two segments: (1) sorting problems into categories "easy" and "hard" and talking about the reasons for sorting, and (2) talking about how problems could be solved. Occasionally a third segment was added by directing a student's attention to the characteristics of the problem (e.g., for 46-19, "Is there a way to make this problem easier?"). When requested, a student was allowed to sort a card into an intermediate category midway between "easy" and "hard" (e.g., "This one is sorta easy and sorta hard"). Interviews lasted 15 to 30 minutes and were video recorded, conducted in students' native language, and transcribed for data analysis.

Participants

The cross-national comparison required students from countries with different school systems and approaches to mathematics education. In

Germany, there is a great emphasis on mental calculation (Krauthausen, 1993) in the elementary grades, with the standard computing algorithms for addition and subtraction introduced in the middle of third grade for three-digit numbers. In contrast, American students are taught the standard addition algorithm in first grade, as soon as they encounter two-digit addition. Similarly, the standard subtraction and multiplication computing algorithms are taught in second and third grades, respectively. Based on this important curriculum difference and results from former research (Selter, 2001), it was expected that second graders would be more flexible than fourth graders, and that German students would be more flexible than American students. Assessing this cross-national discrepancy was a primary motive for selecting project participants.

Sixty-nine elementary students were interviewed, all high and middle achievers selected by their teachers. Pilot testing indicated that low achievers tended to be exclusively unable to judge if a problem was easy or hard and nearly always failed to exhibit reasoning strategies under investigation, which led to their exclusion from this research. The German sample consisted of 19 second-graders and 11 fourth-graders. The American sample consisted of 22 second-graders and 17 fourth-graders. American students attended school in Charlotte, North Carolina, and German students were schooled in Baden-Württemberg. Students came from ten different classrooms (three second-grade and two fourth-grade classrooms in each country).

Results

The methodology and sample generated the expected strategies in students' performance of mental arithmetic. The sorting and reasoning patterns evoked by problem characteristics provided empirically useful indicators of cognitive flexibility in mental arithmetic. In this vein, whenever students relied on number characteristics and numerical relations, they exhibited not only a variety of reasons but also reasons that were well adapted to individual problems. Consequently, we concluded that it is possible to more directly access mental arithmetic than previous research might suggest. Students' use of number patterns and relationships provided a more direct, differentiated, and appropriate way to define and operationalize flexibility in mental arithmetic with regard to cognitive elements that sustain the solution process than alternative approaches measuring elapsed time and solution accuracy or goodness of fit between problem and solution strategy (cf. Star & Newton, 2009; Torbyns et al., 2009; Verschaffel et al., 2009).

Among our findings were the following (Rathgeb-Schnierer & Green 2013, 2015, 2017a, 2017b).

- Problem characteristics were reported twice as often as solution procedures as the basis of student's sorting "easy" versus "hard" problems.

- Problems sorted as “easy” generated a greater variety of flexible reasoning than did problems sorted as “hard.”
- Classrooms can generate different patterns of cognitive flexibility in their students’ reasoning.

The methodology effectively differentiated between strategies that relied on problem characteristics and number relations versus those that depended solely on calculation procedures. As expected, there were also clear patterns that emerged in the analysis of problems typically sorted as “easy” versus those sorted as “hard.”

Phase 2: Reasoning Patterns, Cross-National Comparisons, and Profiles of Flexibility

Primary research interest was the extent to which students exhibited reasoning by problem characteristics, which would reveal a number sense not reducible to memorized facts and computational algorithms (Rathgeb-Schnierer & Green, 2017a, 2017b). For easy problems, student reasoning referred to numerical relations for about one-third of the problems, to number features for just under half of the problems, and to basic facts for about one-fourth of the problems. In this context, reasoning by numerical relations comprises relations between numbers (e.g., range of numbers, double half, sums of ten), relations between problems (e.g., inverse problems, commutativity, related problems), and analogies of tens and ones. Reasoning by number features included special features of the ones (e.g., sums of ten, no regrouping or renaming needed), special numbers (e.g., double digits, numbers close to the next ten), or the size of numbers. Reasoning by basic facts indicates that parts of a problem or a whole problem are known by rote memorization.

In contrast, problems were sorted “hard” by students producing reasoning predominantly based on number features, most often based on features of the numbers in the ones place (four-fifths of all “hard” sortings). For many students, the need to rename for subtraction problems was often a sufficient reason for them to label a problem as “hard.” Ironically, other students who noticed number features just as typically reported the same problem to be “easy.”

To explore patterns of students’ flexible reasoning, we examined both the frequency of flexible reasons and the repertoire of flexible reasons. Such a distinction was important because some students may exhibit a high frequency but limited range of flexible reasons. In this context, some students provided multiple, flexible reasons for their sorting; others provided none. For the entire sample, sorting coded as flexible ranged from 0 to 17 (for 12 problems. Plotted as a histogram, the data reflected a continuous distribution of every possible value between the two extremes (0, 17); more individuals were clustered toward the center of the distribution than toward its extremes.

In a comparison of flexible reasoning *frequencies* exhibited by German and American students, no country difference was found. However, a significant difference was reported for a grade, with fourth-graders producing significantly more flexible reasons than second-graders. No country difference was found (see details in Rathgeb-Schnierer & Green, 2017b).

Regarding the *repertoire* (number of different types) of flexible reasons, individual students ranged from 0 to 11. As with flexible frequencies, the repertoire histogram also reflects a continuous distribution across all possible values between the two extremes (0, 11). The findings regarding the repertoire offer an important check on the results reported for the frequency of flexible reasoning. In a similar vein, a significant difference was found between second and fourth graders, but no country difference was found (see details in Rathgeb-Schnierer & Green, 2017b).

While historical treatments of mental flexibility tended to adhere to the flexible – rigid dichotomy, our sample exhibited cognitive flexibility in mental arithmetic best characterized as continuous rather than bimodal (Rathgeb-Schnierer & Green, 2017b). Moreover, students' cognitive flexibility typically fits one of three profiles along this continuum. Nearly one-third of the students were categorized as flexible (Profile F), and these students exclusively used reasoning by problem characteristics in their sorting. Only a few (less than one-tenth) exhibited extreme rigidity (Profile R), and their approaches were dominated by solution procedure schemes. The majority (just over 60%) displayed some mixture of rigid and flexible reasoning (Profile M). While German and American students exhibited no difference in their cognitive flexibility, fourth-graders typically exhibited more mental flexibility than second-graders.

Phase 3 – Pedagogical Extensions

Baker et al. (2010) have argued that important changes in the content of American mathematics curricula after the mid-1960s reflected an emerging re-conceptualization of children's cognitive capabilities. Specifically, the cognitive revolution in psychology, driven principally by the Piagetian constructivist agenda of how children acquire and use knowledge, supplanted the then-dominant behaviorist agenda in developmental psychology. An important product of that revolution was the subsequent publication of NCTM's (1989) *Curriculum and Evaluation Standards for School Mathematics*, which attempted to reform mathematics education with a new emphasis on conceptual understanding and problem-solving - informed by Piaget's constructivism – with reduced emphasis on rote learning of symbolic facts and computing algorithms (McLeod, 2003). In the wake of these changes, American mathematics education began to "incorporate activities designed specifically to exercise and promote these abilities, including basic reasoning abilities" (Baker et al., 2020, p. 416).

In that historical context, we have attempted to enlarge our understanding of the cognitive elements that motivate and inhibit mental flexibility in solving arithmetic problems. Our later work has sought to extend the research on cognitive flexibility into the classroom (Rathgeb-Schnierer & Green, 2019). One promising development from this area of research has been reported by Schütte (2004) and Rechtsteiner-Merz (2013), who developed a special approach to mathematics education that emphasizes the recognition of problem characteristics and numerical relationships. The approach is called *Zahlenblickschulung*. *Zahlenblickschulung* is a long-term (no pun intended) approach that extends over the entire period of elementary school, and it targets the development of number concepts and the understanding of operations and strategic means (Rechtsteiner & Rathgeb-Schnierer, 2017). The basic principles of this approach are:

- To postpone solving problems in support of focusing on problem characteristics and relations between problems.
- To develop metacognitive competencies by posing cognitively challenging questions to provoke students' thinking and reflection.

The *Zahlenblickschulung* approach underscores and supports the development of *Zahlenblick*, which refers to "the competence to recognize problem characteristics, number patterns, and numerical relations immediately and to use them for solving a problem" (Rechtsteiner & Rathgeb-Schnierer, 2017, p. 2). For elementary students, such competence can be fostered by activities that engage them in sorting and arranging actions that promote their recognition of number patterns, problem characteristics, and relations between numbers and problems. Correct answers to arithmetic problems are not computed during these activities because the focus is on the problem and numerical characteristics. In such exploratory situations, students have opportunities to discover inherent (e.g., construct) numerical structures and relations (Rechtsteiner & Rathgeb-Schnierer, 2017).

There is growing evidence that all students benefit from the *Zahlenblickschulung* approach in developing flexibility in mental calculations (Rechtsteiner-Merz, 2013; Rechtsteiner & Rathgeb-Schnierer, 2017). However, for students who have learning difficulties in mathematics, learning how to attend to problem characteristics and numerical relations is a critical condition for them in developing solution strategies that go beyond simple counting (Rechtsteiner & Rathgeb-Schnierer, 2017). In a similar vein, Rathgeb-Schnierer and Green (2015, 2019) underscore the importance of analyzing student reasoning as a prime indicator of their flexibility in mental arithmetic and as the basis for a child-centered approach to arithmetic pedagogy in elementary classrooms.

The Special Issue on Cognitive Flexibility

Four articles comprise the remainder of this special issue. Each makes a significant contribution to our understanding of some aspect of the research on cognitive flexibility in arithmetic. The order of their presentation is based on our intuition about linear continuity for our readers. In that vein, the articles are presented in the following order of their ideational relationship to primary literature on cognitive flexibility:

- A literature review comparing the effectiveness of teacher- versus student-initiated pedagogies
- A replication study of important elements of cognitive flexibility
- A case study of exemplary teacher actions that promote cognitive flexibility
- A case study of student-to-student interactions that promote cognitive flexibility

A Comparison of Teacher-Led and Student-Centered Instruction

Heinze et al. provide an analysis of previously published empirical research to compare the relative effects of explicit versus implicit teaching aimed at promoting cognitive flexibility. The authors present a review of relevant research studies on students' adaptive use of strategies, an important aspect of cognitive flexibility. The literature leads them to differentiate between explicit (teacher-based) and implicit (child-centered) mathematical pedagogy. In their analysis, they detail elements of each approach in terms of its ideal typology, and they examine the presumptive prerequisite knowledge and skills required to make adaptive use of strategies.

In their presentation, Heinze et al. evaluate five research studies that speak directly to the comparison between explicit and implicit pedagogies. Each study is examined in summary form, but with sufficient detail to identify its essential explicit or implicit approach to student learning.

Three major conclusions are drawn from the results of studies about explicit instruction for the adaptive use of strategies. In addition, contrasting outcomes are described for the implicit instruction approaches. The authors note the relative dearth of research that directly compares explicit versus implicit teaching of flexible arithmetic strategies. Equally important, they argue that much of the published work is severely limited by specificity: to specific samples, using specific strategies, with specific problems. The overarching problem of generalization, a primary goal of modern education has been a significant shortcoming of work in this area.

A Replication of Cognitive Flexibility Among Brazilian Elementary Students

Nunes et al. report on their transcultural replication with Brazilian students of two critical features reported in the Rathgeb-Schnierer and Green (2013, 2015, 2017a, 2017b) research originally conducted on American and German elementary students. First, they attempt to adapt the problem sorting and structured interview methodology for Brazilian second and third graders, whose arithmetic education is less advanced than it is in the United States and Germany. Second, they attempt to use elements of the original research reports to determine the extent to which the profiles of cognitive flexibility can be replicated with Brazilian elementary students. The authors report detailed analyses to establish their case for the importance of replication for Brazilian educational planning and for teaching designed to foster cognitive flexibility.

The importance of research into cognitive flexibility is particularly important for Brazil, whose educational system is far less developed than that of North American or European systems. Brazil needs approaches to teaching that transcend rote memorization and automatic paper and pencil calculation. If Brazil is to progress educationally, then its system and its decision-makers will need to attend to modern research about student competencies, like this report on student cognitive flexibility in mental arithmetic.

Teacher Actions that Promote Cognitive Flexibility

The Serrazina and Rodrigues article is primarily concerned with how teachers can improve their students' cognitive flexibility. They provide here a detailed case study of teacher-directed activities with 26 paired second-grade students engaged in sorting multi-digit addition problems as either "known quickly" or "not known quickly." This task is similar to the Rathgeb-Schnierer and Green (2013, 2015, 2017b) task of sorting problems as "easy" or "hard." Student pairs are asked to explain the reasoning behind their sorting, and they are guided by the teacher's questions to find similarities with other problems and relationships between numbers and addition problems.

The authors describe how the teacher's actions, through questions about similarities in numeric expressions or known facts, can guide students to make new connections they had not previously considered. The teacher's questions, without ever giving students the answers to problems, challenged them to make sense out of and reason about connections and similarities. Ultimately, Serrazina and Rodrigues conclude that the students' cognitive flexibility was improved in such a way that could be generalized to other classroom situations.

Student-to-Student Interactions that Encourage Cognitive Flexibility

Korten presents a design research study that evaluates flexibility in mental calculations in elementary students. The author argues that the development of flexible mental calculation can occur in student-to-student interactive-cooperative learning situations. To test this idea, a teaching-learning arrangement was designed to encourage student communication about problem characteristics and relations.14 pairs of German second and third graders (paired with and without learning difficulties) were observed performing and comparing addition tasks. Sequences of paired interactions were transcribed and analyzed by an interpretative approach Results show specific learning processes and productive moments occurred in every student-to-student pair. Based on the results of this study, a "productive moments" template is recommended for pedagogical planning designed to improve students' cognitive flexibility in mental calculation.

Future Research

Our research program has been based on both theoretical and pragmatic concerns. While we have begun to address important issues attendant to cognitive flexibility in mental arithmetic, important work remains. Using our work as one possible platform, we suggest the following research questions and issues for future research.

1. Given our direct assessment of cognitive flexibility, to what extent is it cognitive flexibility in mental arithmetic naturally expressed at five levels of application: elementary school, middle grades, secondary school, college, and working adults? Answering this question is important because, either flexibility has a developmental arc, or it doesn't; either it has an adult (mature) pay off or not. Having a direct assessment, like our problem sorting task, usable across all five levels of application would suggest possible comparisons not previously available to researchers.
2. Are there measurable performance assets associated with cognitive flexibility as compared to cognitive rigidity? Having a direct measure, it should be easier to design research using cognitive flexibility as a main effect or as a covariate. This area of research has the potential to dramatically expand our understanding of relationships between cognitive flexibility and other mental and educational variables. For example, correlations along the continuum of rigidity-flexibility could be composed with school performance, international test data, and end of grade tests (used mainly in the U. S.).
3. Are some flexible strategies more prevalent than others? That is, do they have more utility and generalizability than others, either across problems or across arithmetic operations? And to what extent are some flexible strategies uniquely situation-specific (e.g., sums or differences equating to zero) versus generalizable?
4. To what extent are our findings generalizable across student populations and arithmetic operations? Are some elements of cognitive flexibility

more generalizable than others? Are some elements of cognitive flexibility more easily taught than others?

These are the types of questions that motivate our interests. As exemplified by the four summaries above, others are pursuing different but important agendas.

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