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Variation Approach on Semiotic Registers/Sets Using Eastern-Western Abaci in Primary Schools

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The paper presents the design and the preliminary feedback of a multimodal tool shaped by the Variation Approach, involving the use of Western, Chinese, and Japanese Abaci and their visualisations during simple computations. The approaches were used to solve different types of tasks. With this purpose we asked to the 60 primary-school students, between the ages of 7 and11 years, to illustrate their solution paths. Based on these preliminary findings, we will show the strengths and weaknesses of the tool for the diagnostic purposes of arithmetic proficiency.

Keywords: Variation approach, multimodality, Abacus as didactical artefact, visualisation, diagrams

An abacus is one of the first mathematical tools developed by human beings to aid in counting. There are several kinds of abaci depending on the culture in which they have been developed. One is the western abacus and its various sub-groups, nine-bead and ten-bead ones, in which each bead counts as one. The Chinese and Japanese ones, or the eastern abaci have beads that have different values using a 5-value bead. In this paper we used several theoretical frameworks, from Eastern and Western traditions, to show the differences in various situations in which they are used. The main purpose of this paper is to show and explain the building of a counting tool to demonstrate the link between arithmetic and pre-algebraic thinking (Brousseau, 1986).

In Bianco and Di Paola (2022) we shared an analysis of the use of different kinds of abaci in the Chinese textbook at a primary level. We designed two workshop activities focused on the use of the Eastern abaci in three Italian classrooms (two V-grade classes and one II-grade class: above 60 students). We plan to show that different kinds of abaci can be very useful to develop a pre-algebraic and pre-symbolic thinking in our students from a new viewpoint.

The first workshop activity is designed for a second-year primary class as an activity to show the given numbers with each task and then to visualise and draw the correspondent schematisation (see Figure 1). The second workshop, for two fifth-grade classes, the task was not just to represent on the paper the static result of the operations involved but all the dynamical path of the operation. The students needed to show the different steps and the role of the different parts in an addition and subtraction problem. While students in both workshops worked on the same problem (Figure 1), the focus and the purpose was different.

Both workshops are structured as multistep activities, inspired by the Chinese Variation Approach (Gu et al., 2004). This powerful tool can be used to show the relationships between tasks and representations (Sun, 2011). We start from a synthesis of the theoretical framework to underline the benefits of these western and eastern tools. This multicultural activity provided the opportunity to discuss theories of different contexts, with a common core.

We denote visualisation as "the recognition of what is mathematically relevant in picture or diagram" (Duval, 2014, p.160). According to Duval (1999), considering that "visualization in mathematics is needed because it displays organization of relations" (p.14-15). Visualisation, which stands in the semiotic regime, can show mathematical operations in its continuous path because "visualization consists in grasping directly the whole configuration of relations and in discriminating what is relevant in it" (Duval, 1999, p.14). We will show how this is relevant to the Variation perspective (Gu et al., 2004).

Diagrams, according to Peirce as cited in Stjernfelt, 2000, a special case of images, which show the likeness between the object and the operation. The visualization stage can be seen before students solve the problem (Dörfler, 2005).

Definitions

The word abacus will be used for all the family of abaci, and each specific kind of abacus will be pointed using the adjective "*eastern*" or "*western*".

The Variation Approach (Gu et al., 2004; Sun, 2011), also called *Bianshi* 变式, as a teaching practice typical in Chinese countries, and encoded based on class observations by Lingyuan Gu in 80s.

The Variation Theory will be the eastern-western theoretical and pedagogical framework of the Variation Approach (Cheng & Lo, 2013; Pang et al., 2017), and of learning in general (Lo & Marton, 2012; Marton et al., 2004).

Framework: A Semiotic Bridge Between East and West

In this section we will briefly recall the theoretical frameworks useful for our purpose. The focus will be on the substantial accordance on the multimodality which each theory approached in a different way.

The framework of this study is based on the work of Duval (2017) in which the emphasis is on the simultaneous use of different semiotic registers (verbal, schematic, graphic, symbolic). To understand a mathematical task the learner needs to deal with many instances and recognize through several different representations of the same object, which can be seen as the "real" mathematical meaning.

This general semiotic perspective can be joined, regarding the principles (Radford, 2008) in the sense of a comparison (Prediger et al., 2008), with another theoretical frame. This can be useful to analyse epistemologically the tasks, as described in the Variation Theory. This is the main tool used for the design of the activities given to the students (Figure 1). In order to include the tasks and their rules we have to break down the semiotic system (Arzarello (2006), and Duval (2017). For the first three columns of the sheet (Figure 1 and Table 1) the notion of representations can be a good starting point (Duval, 2017). According to Vergnaud (2009), a conceptual understanding will be shown in the verbal explanations of tasks for addition.

In addition, Vergnaud (2009) recalls "the development of a conceptual field … through… contrasting situations", it's said that "the function of *schemes* […] is both to describe ordinary ways of doing, for situations already mastered, and give hints on how to tackle new situations" (pp.85 & 88). This aligns with Radford (2008), on the Variation Theory in which is stated that "there can be no discernment without experienced difference, and there can be no experienced difference without a simultaneous experience of at least two things that differ. So discernment, simultaneity, and variation go together" (Lo & Marton, 2012, p. 10) because "awareness of a single feature cannot exist without the awareness of differences (variation) between features" (Lo & Marton, 2012, p. 10). This constructivist view enhances the characteristics of each situation and their perception as something, new or known, in comparison with future or past (operational) schemas of acting useful for the learner: "although teachers cannot make their students learn, they can help make learning possible. […] There are certain conditions necessary for learning. To discern the *critical features* of novel situations, the learner must "experience for themselves" certain patterns of *variation* and *invariance* of these features" (Lo & Marton, 2012, p. 11-12). The balance between variation changes at every step (fusion, separation, contrast, generalization), the Variation Theory frame a learning development based on perceptions and on the interaction between successful schemas (Marton et al., 2004). This is what is said by Vergnaud: "on the one hand, a *scheme* is the invariant organization of activity for a certain class of situations; on the other hand, its analytic definition must contain open concepts and possibilities of inference" (Vergnaud, 2009, p.85). This is why Variation, as an approach as well as a theory, can be acknowledged as the bridge between the two cornerstones of the Chinese mathematics education: the Two Basis. The gap between Theory- and Computations-driven-learning is dependent on variation, on both in concepts and procedures. The constructivist interpretation (Vygotsky, 1978) is indeed fundamental to the frame of the Chinese Variation Approach (Gu et al., 2004). The framework described above, is a comparison (Prediger et al., 2008) between Eastern and Western theories, and is rooted in the common constructivist frame. The role of signs in these theories all give emphasis to the step-by-step construction of knowledge (Gu et al., 2004; Radford, 2003; Sáenz-Ludlow & Kadunz, 2016). See Table 2.

Methodology

We will describe below the composition of the Nine Variations on Abaci activity and how it fits in the given framework. We will also provide a short background on the design of the workshops.

Design

According to the Variation Theory (Marton et al., 2004), on an epistemological level "to see something in a powerful way amounts to discerning its critical features and focusing them simultaneously. Whether or not the learners can discern critical features and learn to discern critical features in a certain situation depends on what varies and what is invariant in that situation; the reason being that we can only discern that which varies" (Gu et al., 2004, p. 334). From an educational perspective "the theory suggests is that for any specific capability that we might wish to develop in students, they must experience a certain pattern of variation/invariance in order to develop that capability" (ibid., p. 336).

The activity presented in Figure 1 and Table 1 shows a structure rooted in the Variation Approach (as a Chinese educational practice) (Gu et al., 2004) and Variation Theory (Pang et al., 2017) (see Table 2). The switch between semiotic registers (language, symbols, schematisations of artefacts), and problems, can be seen as a simultaneous presentation of different problems. According to Sun (2011), this is an example of Multiple Problems One Solution, for the cross-column-reading. In the case of One Problem Multiple Solutions and One Problem Multiple Changes, we have cross-row-reading, in which the symmetry between addition and subtraction is shown. The cross-column-moving is, according to Gu et al. (2004), on a conceptual level, while the cross-row-moving is on a procedural one.

Figure 1

Nine Variations on Abacus Used During the Workshop Activities

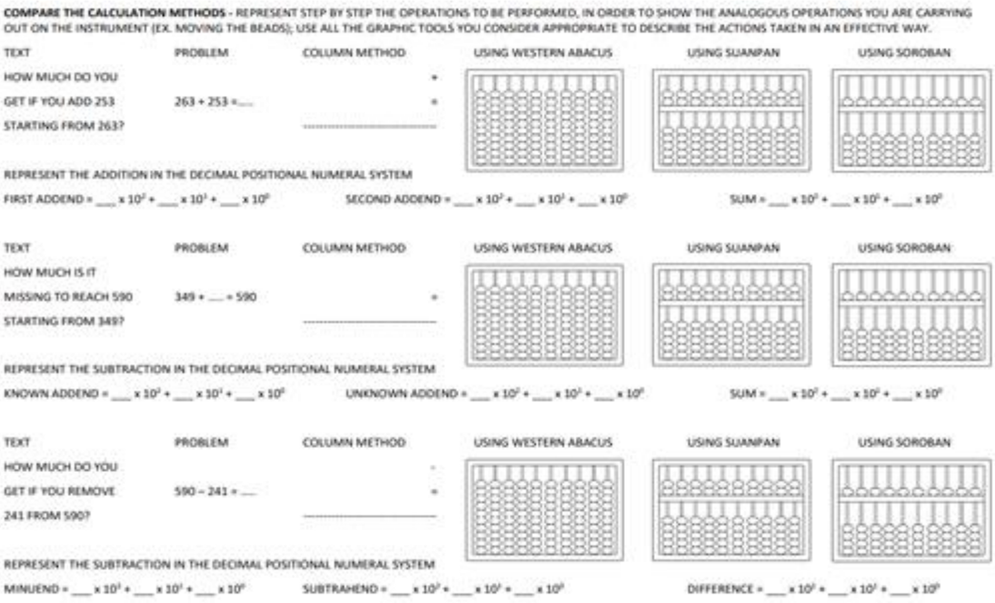


Table 1

The Whole Activity Including the Corresponding Theoretical Frameworks

<table>
<tr><td>According to Figure 1</td><td>Text – column one in Figure 1</td><td>Problem – column two in Figure 1</td><td>Column method – column three in Figure 1</td><td>Using Western abacus – column four in Figure 1</td><td>Using Suanpan – column five in Figure 1</td><td>Using Soroban – column six in Figure 1</td><td></td><td></td><td></td></tr>
<tr><td>Columns’ role</td><td>Text of the problem</td><td>Symbolical presentation of the task</td><td>Space to do the column operations</td><td>Graphic sketch of the Western abacus</td><td>Graphic sketch of the Chinese abacus</td><td>Graphic sketch of the Japanese abacus</td><td>The Western abacus (artefact)</td><td>The Chinese abacus (artefact)</td><td>The Japanese abacus (artefact)</td></tr>
<tr><td rowspan="2">Theoretical Frameworks</td><td>Vergnaud (2009)</td><td colspan="2">Duval (1999, 2014, 2017)</td><td colspan="3">Sáenz-Ludlow & Kadunz (2016)
Stjernfelt (2000)</td><td colspan="3">Radford (2003)
Bartolini Bussi & Mariotti (2008)</td></tr>
<tr><td colspan="9">Gu, Huang & Marton (2004); Lo & Marton (2012); Sun (2011); Arzarello (2006)</td></tr>
</table>

Moving on a fixed row there is a sequence, first translation of the count needed into the symbolic register, then a semi-structured space where pupils can do the counts, and finally a pictorial schematisation of three kinds of abacai.

In each of these parts, moving on a fixed column, the students work on numbers and diagrams. Looking at Table 1, there are three other columns linked to the real artefacts. Each artefact differs in its ways of use (Arzarello, 2006).

Reading row-wise the Vergnaud's frame becomes central: starting from the first row where the problem is a static addition in which are known both addends (263 + 253 = _), the second row becomes a dynamical addition (349 + _ = 590), where an addend is missing, finally in the third row there is a static subtraction situation (590 – 241 = _).

The reason why we will give both verbal and symbolic as well as pictorial registers is to scaffold from what is known by the students to which they need to know.

Variation Theory and Multimodal Tasks

Based on Duval (2014) we analyze the corresponding visualizations given by students, with the aim to improve the design of the multimodal and variational tool proposed. Visualization here can be a dot, a picture, or a detailed diagram representing a mathematical process (Duval, 1999). In all three, there is a link to the original configuration shaped/modelized, so stays the role of intuition due the new and open-problem given, in which the real difficulty is on an expressive/communicative level (Sáenz-Ludlow & Kadunz, 2016).

The schematisations of the abaci, can be looked at as a picture or is can be used as a mathematical tool on which students can operate. The combination of both, a heuristic one and a more formal one (see Duval, 2014), is demonstrated in the two workshops' tasks. First by representing the result, and then reviewing the path followed to show how the solution was reached. In the first case the schematisations can be seen just as a graphic support. In the second case, to help solve the task. Students had to:

- begin with the concrete stage of the physical artifact;
- move to the representation stage by creating a diagram;
- and reasoning using the representation to solve the problem.

By moving through this sequence, students can understand how the tool works to model mathematics and provide opportunities for algebraic thinking with beginning number concepts. In this way this activity can be seen as a pre-symbolic but algebraic task, (Bianco & Di Paola, 2022).

Furthermore, the second task, for older students, is not just an exercise but a problem in which there are no correct answers. As briefly sketched in Bianco and Di Paola (2022) the Eastern tradition emphasizes the strategic role of chosen exercises as a part of a whole problem, with the aim to build a *space of relations* (Sun, 2011), in opposition to a Western practice in which there is a huge amount of very easy exercises driven to a very specific skill (Gu et al., 2004; Watson & Mason, 2006).

Design of the Workshop Activities

During the workshops in the three classrooms, an Eastern kind of Abacus (Chinese or Japanese one) was given to each group. The classes comprised of 4/5 groups, each with 4/5 students. One artifact was given to each group. One of the two fifth-grade classes were already familiar with the abacus. However, they still struggled to use the tool. The tasks were more pre-algebraic (Cai & Knuth, 2011) in nature, focusing on the mathematical structure and the rules of the instrument. The second fifth-grade class was led by a teacher who had experience adapting to different situations An activity design such as the one described here can help to identify and overcome misconceptions and create a new set of solutions (Sun, 2011). Here the Variation Theory (Kullberg et al., 2017), and *Bianshi* or Variation Approach (Gu et al., 2004), can be seen.

Research Question and Purpose

The research question aimed to answer the following:

How can variation (Gu et al., 2004) and multimodal-semiotic (Arzarello, 2006) approaches be used together (Prediger et al., 2008) to build a tool useful to identify and overcome challenges (Brousseau, 1986) in arithmetic tasks (on positional and decimal numeral system) and to develop pre-algebraic proficiencies, at a primary level?

The purpose of this research was to show the use of an abacus in developing algebraic thinking. This allowed students the opportunity to think mathematically using visualization. In addition, it's often necessary to care about or create hybrid registers, re-interpreting socio-cultural conventions, even beyond what already done in classroom (Brousseau, 1986).

A Preliminary Analysis of the Results

From the Use of the Abacus to the Visualisation of an Open Problem

By examining the difficulties or challenges students had, we can identify ways to strengthen future workshops. The first challenge was around the use of colour. In Italy it's quite common to have units that are blue, tens red, and hundreds green (as in Figure 3c 1). This can be useful for the first years, but, already since the second part of the II-grade, it becomes a misconception. During the workshop, younger students didn't use colours at all or used too many of them incorrectly. Some students (25%), rather than paying attention to each digit, focused more on the on the two addends each as a whole.

The second challenge had to do with choosing a direction. Students struggled to choose a direction to move the beds and caused conflict among the groups. Students also didn't "align" the addends on the abacus, as in the column method, and drew the representations simply where they landed.

The third challenge was around the role teachers played. Some teachers were not willing to use the tool and their students were not motivated to illustrate their solutions.

A fourth challenge was with the relationships between numbers and position. To get an algebraic understanding of the structure of the abacus, the Eastern abacus work was challenging for all students who were not familiar with the tool. During the explorative phase on the activities, they tried to give the same values to each bead and then started to search for beads representing tens by moving around the abacus. Some of the students gave to each of the lower beads the value of two, making counting possible, even if a bit too cumbersome.

The final challenge was with the tasks for older students. They could perform the task orally and could move through the stages without needing the tool to help with a solution strategy.

Using a Multimodal Approach

A real help to overcome the difficulties above was the use of the abacus to simulate and understand the problems, and working in groups to solve the tasks. The activity encouraged multiple ways to solve the problem by breaking the rules and going beyond the what the assignments were asking. Students were encouraged to reshape the tasks using diagrams. This could be seen throughout the workshops where many students demonstrated work using graphic abaci. The students did this by using conversions to shift between the different kind of codes and abaci. They were all able to use different representation to demonstrate solutions.

Students created detailed illustrations of the coloured beads and using words to explain the meaning of the colouring. Students chose the direction and the value of each bead to *express* their strategies. Several students discussed the diagrammatic parts of the illustrations showing the operations in action.

Table 2

On the Relations Between Western and Eastern Theories

Method of analysis	Prediger, Bikner-Ahsbahs & Arzarello (2008) Radford (2008)	
	Western Theories	Eastern Theories
On Semiotics	Duval (2017)	*Bianshi* or Variation Approach: Gu, Huang & Marton (2004); Sun (2011)
On Learning	Vergnaud (2009)	Variation Theory: Marton, Runesson & Tsui (2004); Lo & Marton (2012); Pang, Bao & Ki (2017)

Figure 2 a

Crosses and Carries (During Addition)

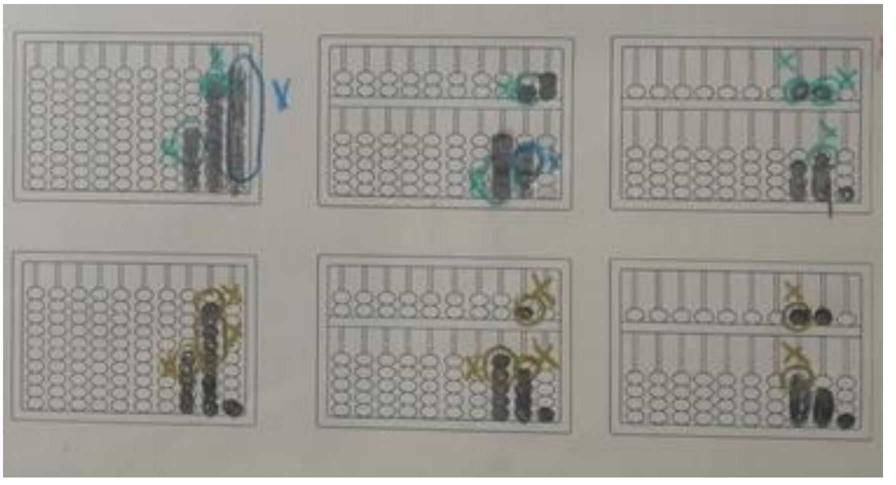


Figure 2 b (1)
Arrows and Crosses (During Addition

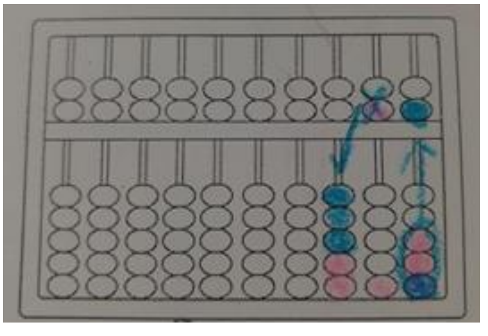


Figure 2 b (2)
Arrows and Crosses (During Addition)

Note. Written in Italian: "con Suanpan" which means "using Suanpan" and "togliere" which means "put away/delete".

Figure 2 b (3)
Arrows and Coloured Overflow (During Addition)

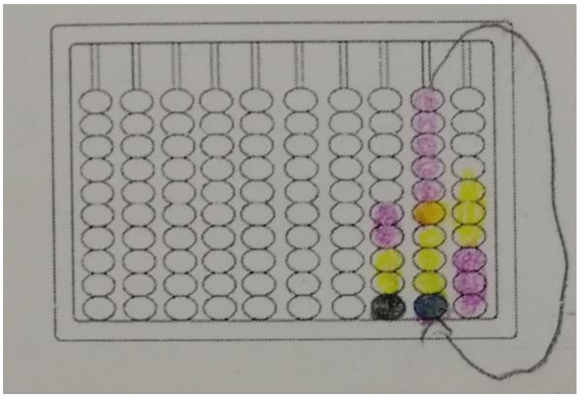


Figure 2 c (1)

Erasing and Arrowing (During Addition), Even Using Conventional Colours

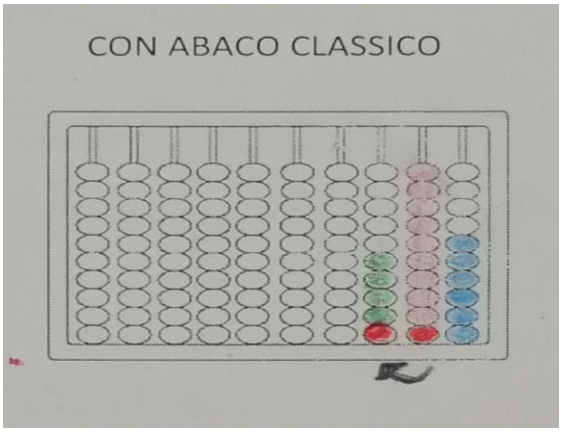


Figure 2 c (2)
Erasing and Subtracting (during Subtraction)

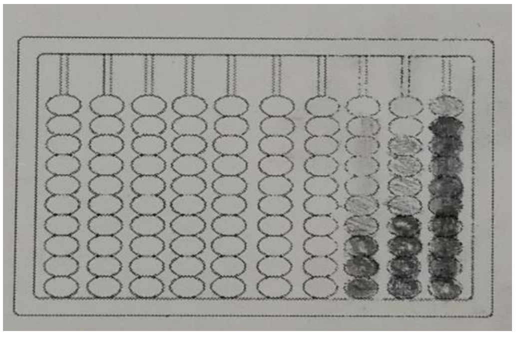


Figure 2 d
Supplementary Beads

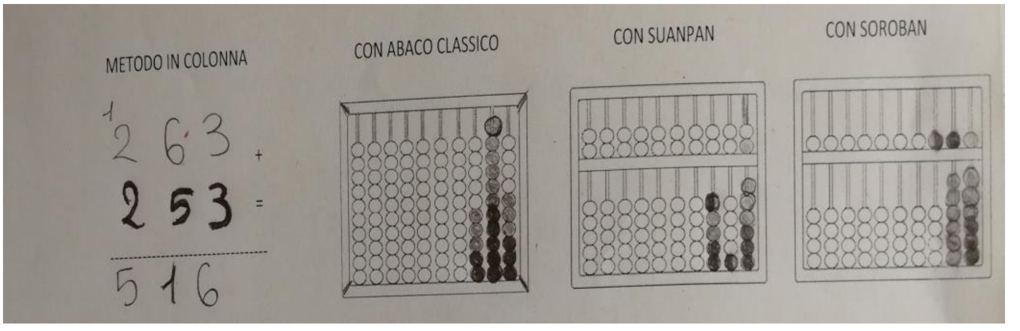


Figure 3 shows a whole resolution by two bright students who collaborated, in which we can see very deep ideas: the use of a balanced amount of colours, including special colours for the carry, the split of a (five-value) bead in its components (2+2+1, 2+3, etc.), crosses/traits to delete beads.

For both, the notation of a number as sum of powers of ten was not already known so they passed through the familiar decomposition in hundreds, tens, units to get a more abstract writing of the numbers involved: $516 = 5 \cdot 100 + 1 \cdot 10 + 6 \cdot 1 = 5 \cdot 10^2 + 1 \cdot 10^1 + 6 \cdot 10^0$.

Figure 3 a
On the Top "[la pallina] usa tutt[i]e due [i] colori"

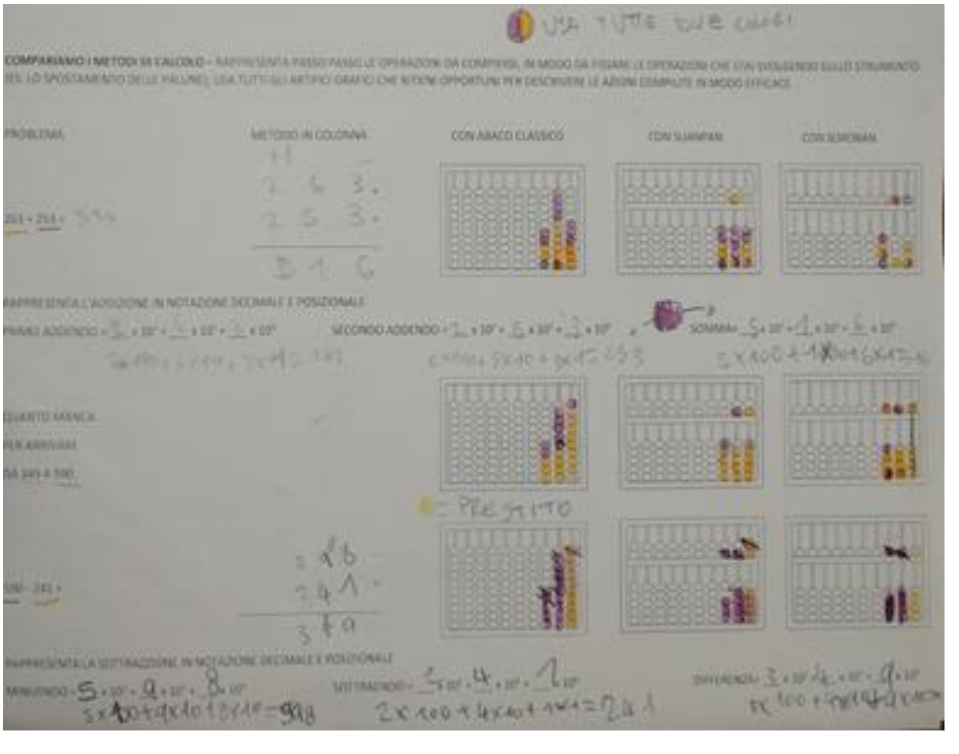
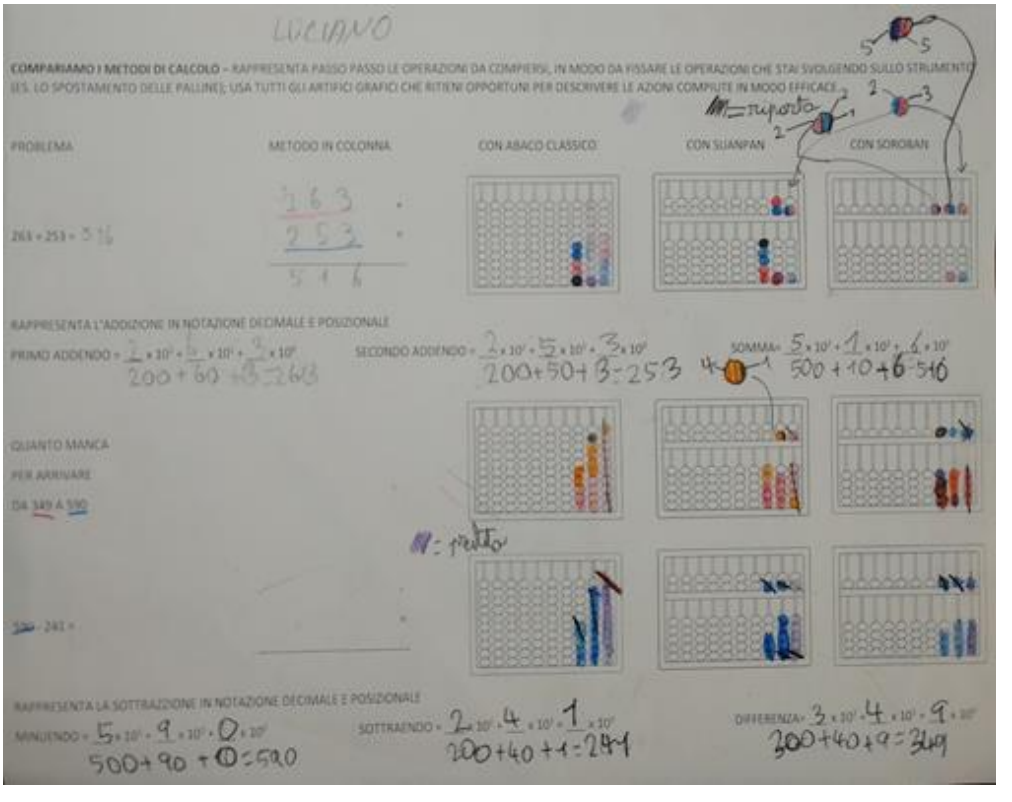


Figure 3 a demonstrates on the top "[la pallina] usa tutt[i]e due [i] colori" which means "[the bead] uses both the colours"; at the bottom "prestito", next to a yellow sketch, means "carry" backwards, during subtraction.

Figure 3 b shows On the Top "Riporto", next to a darken sketch, which means "carry"; the same as at the bottom where "prestito", next to a violet sketch, means "carry" backwards, during subtraction

Figure 3 b

On the Top "Riporto"



Conclusions and Future Work

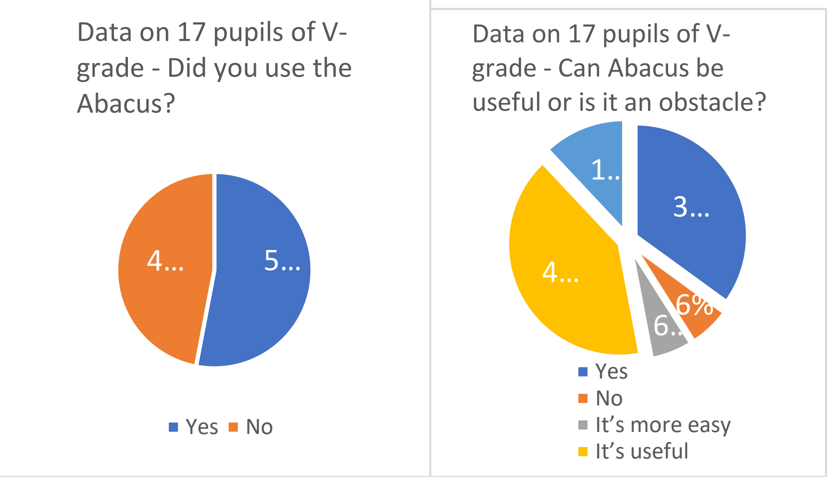
In conclusion, a synthesis on the results presented in Table 3 and Figure 4, based on a questionnaire given to one of the two V-grade class. Half of pupils didn't use at all any kind of Eastern abacus during the activity (Figure 4); but students did find the tool a meaningful way to demonstrate their solutions (see Figure 5). This can be seen as a first step towards a using hands on activities to introduce addition and subtraction.

Table 3

Data from a Questionnaire on 17 Pupils of a V-Grade Class

Data on 17 pupils of V-grade	Most efficient and fast	Slowest	Easiest	Most difficult to use	Used at first	To introduce addition	To introduce subtraction
Western abacus	94%		94%		88%	47%	64%
Suanpan	6%	47%		71%		12%	12%
Soroban		53%	6%	29%		29%	12%
Column method					6%	12%	6%
No answer					6%		6%

Figure 4
Data from a Questionnaire on 17 Pupils of a V-Grade Class



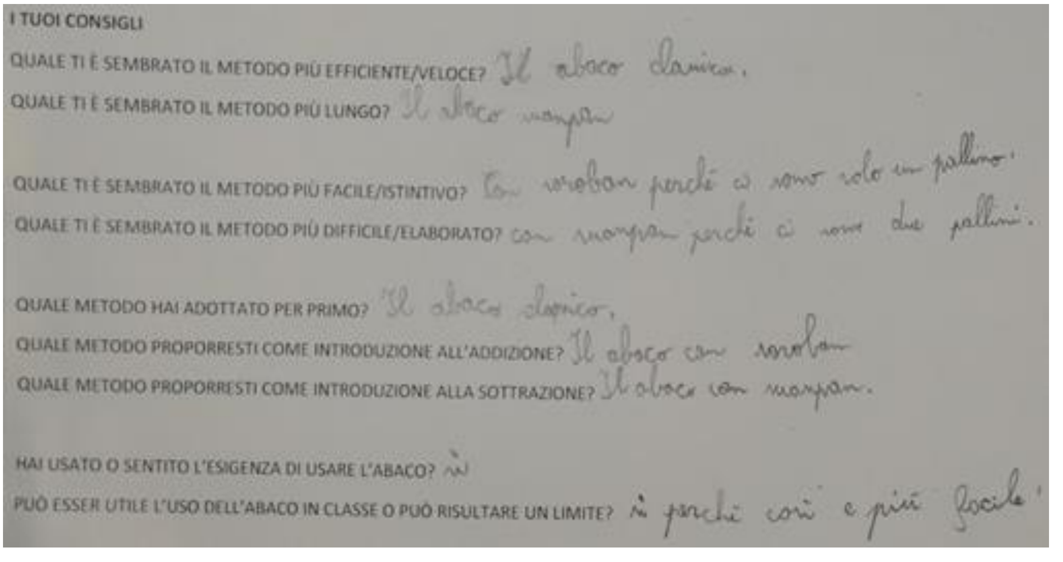
Half of pupils didn't use at all any kind of Eastern abacus during the activity (Figure 4); but students did find the tool a meaningful way to demonstrate their solutions (see Figure 5). This can be seen as a first step towards a using hands on activities to introduce addition and subtraction.

Most students found the Western abacus more efficient, and easier to use than the Eastern ones (Table 2). This may show a cultural bias linked with didactical habits. More work with the Eastern abacus is needed.

Unexpectedly, there was an easy transition between the Chinese and the Japanese abacus, with many students preferring to work with the Soroban, over the Suanpan. This may have been because it has fewer elements to manage. However, one students noted (see Figure 5) that subtractions were easier to do on the Suanpan.

Figure 5 shows students' replies to the Questionnaire with the following questions:
Which method did you find more efficient/fast? The Western abacus;
Which method did you find slower? The Suanpan;
Which method did you find more easy/instinctive? The Soroban, because there is just one [5-value] bead;
Which method did you find more though/tricky? The Suanpan, because there are two [5-value] beads;
Which method did you use at first? The Western abacus;
Which method would you use to introduce addition? The Soroban;
Which method would you use to introduce subtraction? The Suanpan;
Did you use and feel the need to use the Abacus? Yes;
Can be useful or not, the use of the Abacus in classroom? It's useful, in this way it's easier [doing maths].

Figure 5
The Student V.'s Replies to the Questionnaire


I TUOI CONSIGLI
QUALE TI È SEMBRATO IL METODO PIÙ EFFICIENTE/VELOCE? Il abaco classico.
QUALE TI È SEMBRATO IL METODO PIÙ LUNGO? Il abaco suanpan
QUALE TI È SEMBRATO IL METODO PIÙ FACILE/ISTINTIVO? Con soroban perché ci sono solo un pallino.
QUALE TI È SEMBRATO IL METODO PIÙ DIFFICILE/ELABORATO? Con suanpan perché ci sono due pallini.
QUALE METODO HAI ADOTTATO PER PRIMO? Il abaco classico,
QUALE METODO PROPORRESTI COME INTRODUZIONE ALL'ADDIZIONE? Il abaco con soroban
QUALE METODO PROPORRESTI COME INTRODUZIONE ALLA SOTTRAZIONE? Il abaco con suanpan.
HAI USATO O SENTITO L'ESIGENZA DI USARE L'ABACO? sì
PUÒ ESSER UTILE L'USO DELL'ABACO IN CLASSE O PUÒ RISULTARE UN LIMITE? sì perché così è più facile!

Being able to draw was very useful for the younger students, and in the next trials of this workshop will be presented not just during the exploration phase but before any schematisation and mathematical formalisations. This allowed students to do counts using the abacus by themselves rather than conventional mathematics operations (column 4 of Table 1). From these drawings, we could get some insight into students' understanding of the various tasks.

The above results highlighted some weaknesses of the design, including not enough space, and the strict order of the semiotic sets, as well as some characteristic behaviours including the use of colours and orientations, and infrequent use of the instrument.

In this perspective the difficulties and challenges identified can be very useful in the design of future workshops on the use of visualizations (Duval, 1999) in the daily practice of mathematics practice and teaching.

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References

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latinoamericana de Investigación en Matemática Educativa, núm. Esp,* 2006, pp. 267-299.

Bartolini Bussi, M.G., & Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom. Artifacts and signs after a Vygotskian perspective. In L English, M. Bartolini, G. Jones, R. Lesh, B. Sriraman, & D. Tirosh, (Eds.), *Handbook of International research in Mathematics education* (pp. 746-783). New York: Routledge Taylor & Francis Group.

Bianco, G., & Di Paola, B. (2022). Calculus artefacts in Chinese textbooks: Variational approaches with prospective primary teachers. *Journal Of Mathematics Education*, *15*(2), 51-69. https://doi.org/10.26711/007577152790094

Brousseau, G. (1986). Fondements et méthodes de la didactique des mathématiques. *Recherches en Didactique des Mathématiques*, *7*(2), 33–115.

Cai, J., & Knuth, E. (2011). Early algebraization. A global dialogue from multiple perspectives, Springer. https://doi.org/10.1007/978-3-642-17735-4

Cheng, E. C., & Lo, M. L. (2013). Learning study: Its origins, operationalisation, and implications. *OECD Education Working Papers, No. 94,* OECD Publishing, Paris. https://doi.org/10.1787/5k3wjp0s959p-en

Duval, R. (1999). Representations, vision and visualization: Cognitive functions in mathematical thinking. Basic Issues for Learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 3–26), Morelos, Mexico. Columbus: ERIC Clearinghouse for Science, Mathematics, & Environmental Education.

Duval, R. (2014). Commentary: Linking epistemology and semio-cognitive modeling in visualization, *ZDM Mathematics Education* (2014) *46* (1), 159–170. https://doi.org/10.1007/s11858-013-0565-8

Duval, R. (2017). Understanding the mathematical way of thinking – The Registers of semiotic representations, Edited by Campos, T. M.M.; Springer. https://doi.org/10.1007/978-3-319-56910-9

Dörfler, W. (2005). Diagrammatic Thinking affordances and constraints. In M.H. Hoffmann, J. Lenhard, F. Seeger (Eds.), *Activity and sign: grounding mathematics education*. Springer, Boston, pp. 57-66. https://doi.org/10.1007/0-387-24270-8_6

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: a Chinese way of promoting effective mathematics learning. In L. Fan, N. Wong, J. Cai, S. Li (Eds.), *How Chinese learn mathematics: perspectives from insiders*. World Scientific (Series on Mathematics Education; 1) pp. 309-347. https://doi.org/10.1142/9789812562241_0012

Kullberg, A., Runesson Kempe, U., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM Mathematics Education*, *49*, pp. 559-569. https://doi.org/10.1007/s11858-017-0858-4

Lo, M. L., & Marton, F. (2012). Towards a science of the art of teaching Using variation theory as a guiding principle of pedagogical design. *International Journal for Lesson and Learning Studies*, *1*(1), pp. 7-22. https://doi.org/10.1108/20468251211179678

Marton, F., Runesson, U., & Tsui, A.B.M. (2004). The Space of learning. In F. Marton, & A.B.M. Tsui (2004). *Classroom Discourse and the Space of Learning*. Lawrence Erlbaum Associates publishers, pp. 3-40. https://doi.org/10.4324/9781410609762

Pang, M. F., Bao, J., & Ki, W. W. (2017). ‘Bianshi’ and the variation theory of learning, Illustrating two frameworks of variation and invariance in the teaching of mathematics. In R. Huang, & Y. Li (Eds), *Teaching and learning mathematics through variation. Confucian Heritage Meets Western Theories*, Sense Publishers, pp.43-67. https://doi.org/10.1007/978-94-6300-782-5_3

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework, *ZDM Mathematics Education*, *40*, pp.165–178. https://doi.org/10.1007/S11858-008-0086-Z

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students’ types of generalization, *Mathematical Thinking and Learning*, *5*(1), 37–70. https://doi.org/10.1207/S15327833MTL0501_02

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *Zentralblatt für Didaktik der Mathematik – The*

International Journal on Mathematics Education, 40 (2), 317–327. https://doi.org/10.1007/s11858-008-0090-3

Sáenz-Ludlow, A., & Kadunz, G. (eds) (2016). *Semiotics as a tool for learning mathematics how to describe the construction, Visualisation, and Communication of Mathematical Concepts*, SensePublishers Rotterdam. https://doi.org/10.1007/978-94-6300-337-7

Stjernfelt, F. (2000). Diagrams as centerpiece of a Peircean Epistemology. *Transactions of the Charles S. Peirce Society*, *36*(3), pp. 357–384.

Sun, X. (2011). An insider's perspective: "variation problems" and their cultural grounds in Chinese curriculum practice. *Journal of Mathematics Education*, *4*(1), 101-114.

Vergnaud G. (2009). The theory of conceptual fields, *Human Development*, 2009; *52*(2), 83–94. https://doi.org/10.1159/000202727

Vygotsky, L. S., (1978). Mind in society: Development of higher psychological processes. Edited by Cole, M., Jolm-Steiner, V., Scribner, S., Souberman, E.; Harvard University Press. https://doi.org/10.2307/j.ctvjf9vz4

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: using variation to structure sense-making. *Mathematics Thinking and Learning*, *8*(2), 91–111. https://doi.org/10.1207/s15327833mtl0802_1

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