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Special Education Teachers' Knowledge of Fraction Division

Hsiu-fei Lee
National Taitung University
Shuhua An
California State University, Long Beach
Jane Lo
Western Michigan University

According to research, fraction division remains a topic of great difficulty for many students as well as teachers. In addition, it takes longer for special needs students to understand fractions, especially fraction division. Special education teachers are the main source for providing remediation programs for students with difficulties in learning mathematics. Teachers must know the subject they teach. Most current research on fraction division studied prospective teachers in regular education. This article integrates three important aspects that were understudied in the past in mathematics education: special education teachers, in-service teachers, and fraction division in special education. This study is significant because it investigated both procedural and conceptual knowledge in fraction division among Taiwanese special education in-service teachers. The teachers were given a written test consisting of 10 mathematical problems. This article contains in-depth analyses of one problem (1¾÷½) from the test in two aspects. The first involves procedural knowledge of solving the arithmetic problem. The second requires conceptual knowledge based on posing a word problem for the fraction division question that was solved. Most teachers possess adequate procedural knowledge but lack a solid understanding of fraction division. The models and errors of the word problem types are further analyzed. Implications for future research and instruction on fraction division and/or for special education teachers are included in the discussions.

Keywords: Fraction division, special education teacher, procedural and conceptual knowledge, problem posing

Taiwan has the high-performing in mathematics in international assessments, such as TIMSS. Unfortunately, the gaps between the top and low students have been ranked among the top as well. The percentage of Taiwanese 4th graders scored on the low and intermediate benchmarks are higher than the international average in TIMSS 2019. This is an alarming sign of the challenge to education equity in mathematics education. Many students continue to struggle to learn mathematics, especially students with disabilities, regardless of two decades of educational reforms in mathematics education in Taiwan

(Chang, Lee, & Yen, 2019). With time in schooling, the gaps between students with and without disabilities enlarge in Taiwan The same problem exists in other countries too. For example, the gaps between students with disabilities and without disabilities who performed below the basic level were 36% and 48% differences in the 4^{th} and 8^{th} grades (i.e., 51% vs. 15%, and 74% vs. 26% between two groups in the 4^{th} and 8^{th} grades, respectively) based on the report of the National Assessment for Educational Progress (NAEP, 2019). Students struggling with mathematics deserve our attention in mathematics education.

Fractions are one of the topics in school mathematics that many students have difficulties mastering, of which phenomenon is more apparent for students with disabilities. For example, in the U.S., most 4th-grade students failed to identify how many fourths are in a whole, and 68% of students were not able to determine if a fraction was greater than, less than, or equal to another fraction number (NAEP, 2017). Moreover, students with disabilities 84% and 91% in the 4th and 8th grades, respectively, performed below grade level in mathematics, particularly when solving problems with fractions (NAEP, 2017). An international study comparing students in the U.S. and China on fraction concepts and procedures found that national differences were the largest among low-achieving American children; the differences between low-achieving Chinese and U.S. students' fraction arithmetic accuracy were larger (76% vs. 15% in 6^{th} and 78% vs. 20% in 8^{th} grades, respectively) than those of their typically achieving peers (100% vs. 52% in 6^{th} grade, and 100% vs. 80% in 8^{th} grade, respectively (Bailey et al., 2015). To summarize, research shows that students with mathematics difficulties or disabilities have lower fractions achievement (Hecht & Vagi, 2010; Resnick et al., 2016), and make slower gains (Bailey et al., 2015, Jordan et al., 2016; Jordan et al., 2019) than their typically achieving peers. However, as a content area, fractions are understudied when compared to the whole number research in special education (Jordan et al., 2019).

Fractions are not only difficult for students to learn and also challenging for teachers to teach; particularly fraction division is the most complicated operation with complex concepts in the algorithm (Ma, 2010) and it serves as the foundation for a higher level of mathematical learning and reasoning, such as algebra (Lo & Luo, 2012). Although this topic has been widely studied in the past three decades in the context of teacher education and development. The results indicate that many teachers still lack a strong conceptual understanding of fraction division (Adu-Gyamfi et al., 2019; Ball, 1990; Lin et al., 2013; Lo & Luo, 2012; Ma, 2010; Ristiana et al., 2021; Stohlmann et al., 2019; Tobias, 2013; Wu & An, 2008). It is expected that teachers must know the subject they teach. That is, teachers need to possess a 'substantive knowledge of mathematics' as Ball (1990) proclaimed a long time ago. A review paper by Olanoff et al. (2014) indicates that current research on fraction division mostly studies prospective teachers in regular education mentioned above, and many students in our schools struggle with learning mathematics, particularly fraction division. Special education teachers (SETs) are the main source for providing remediation lessons for students with difficulties in learning mathematics. If SETs lack an understanding of fraction division, it will put the special needs students who are already in a vulnerable state in mathematics education into an even more marginalized learning context. That is putting these special needs students into a double disadvantaged state, which Pallia (2022) names as 'at the margin of margins' (Pallia, 2022, p.202, as cited in Annamma et al., 2022). Research studies show that problem posing (PP) in mathematics education has many benefits in learning mathematics (Leung & Carbone, 2013; Ma, 2010; see Osana & Pelczer, 2015, for a review of papers from 1990-2012 on this topic; Xie & Masingila, 2017; Yao et al., 2021). However, creating and solving real world related application problems is a challenge task for students and teachers (Wu & An, 2016, 2017). In An and Wu's study (2017), of all word problems developed by the 3rd graders, 50% of word problems were correct. About 23% of students could not come up with a word problem.

All the unsettling facts described in the above research indicate that equity in special education and/or mathematics education deserves our attention.

This study integrates three important aspects that were understudied in the past in mathematics education: special education teachers, in-service teachers, and fraction division in special education. This study is significant in that it investigated both procedural and conceptual knowledge of Taiwanese in-service SETs on fraction division. The goal is to understand the procedural and conceptual knowledge of fraction division of Taiwanese SETs with no intentions for cross-national comparison. The research questions are as follows:

1. What is the performance of procedural knowledge of fraction division of Taiwanese SETs?
2. What is the performance of conceptual knowledge of fraction division of Taiwanese SETs via problem posing?
3. What models are used and misconceptions involved by Taiwanese SETs when posing their word problems?
4. What are the topics adopted by Taiwanese SETs to pose their word problems?

Literature Review

Teachers' Knowledge of Fraction Division

It is expected that teachers should know the subject they teach. In the case of mathematics, teachers should have a solid knowledge of mathematics because research has shown that there is a close relationship between teachers' content knowledge and student performance in mathematics (Faulkner & Cain, 2013; Kutub & Wijayanti, 2009). Ball (1990) named it the 'substantive knowledge of mathematics' and Ma (2010) as 'profound understanding of fundamental mathematics' (PUFM). This is shown that both procedural and conceptual knowledge are interconnected, and the teachers can flexibly come up with different representations or stories to explain the same topic, such as fraction division as the Chinese teachers in Ma's study (2010). Thus, it is important to study the mathematical knowledge of SETs since they are the major stakeholders for students with special needs in learning mathematics.

Rational number arithmetic, such as fractions, is crucial for later mathematics achievement, developing abstract thinking as well as succeeding in many occupations. However, fractions, particularly fraction division, is a difficult topic in school mathematics, which applies to students and both in-service and prospective teachers (PTs) (Ma, 2010; Olanoff et al., 2014; Ristiana et al., 2021; Stohlmann et al., 2019). It is because it is the most complicated operation with complex numbers in the algorithm (Ma, 2010). The calculation procedures are not the same as those of whole numbers, and conceptually fractions consist of multiple meanings. As a result, not only school children but also PTs have difficulties in figuring out the procedures for calculating the fraction division problem and understanding its multiple meanings (Lo & Luo, 2012; Tobias, 2013). Even the Taiwanese PTs who were highly proficient in elementary and middle school mathematics, when they were assessed through word problems or pictorial diagrams, they also found challenging for them to demonstrate their conceptual knowledge of fraction division (Lin, et al., 2013; Lo & Luo, 2012). These findings suggest that the relationship between their procedural and conceptual knowledge of fraction division was not solid.

Olanoff et al.'s (2014) extensive review of the literature from the 90s to 2013 on elementary PTs' content knowledge of fractions indicates that these PTs possess the procedural skills to calculate fraction algorithms, but lack fraction number sense and knowledge to understand the meanings behind the procedures or why procedures work. Unfortunately, more recent studies of PTs show that this gloomy picture has not changed much (Adu-Gyamfi et al., 2019; Alenazi, 2016; Hohensee & Jansen, 2017; Ristiana et al., 2021; 2014; Stohlmann et al., 2019; Van Steenbrugge et al., Whitehead & Walkowiak, 2017). Many PTs still make mistakes in fraction algorithms and have great difficulties in explaining why the procedures work, particularly for the 'flip and multiply' strategy for fraction division. It is even more challenging for the majority of PTs to come up with a real-life word problem for the fraction division equation.

Lee and Lin (2016) reviewed the literature on numeracy development of teacher education in mathematics published in English and found that most research studied PTs rather than in-service teachers. It is the case for fraction division too. This paper concerns in-service teachers. Now let us learn about in-service teachers' knowledge of fraction divisions. Several cross-national studies show that Chinese in-service teachers possessed a more solid knowledge of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) than their American peers for fraction divisions (Li & Huang, 2008; Ma, 2010). Contrary to the solid content knowledge of Chinese in-service teachers in Ma's study (2010), fraction division remains challenging for many in-service teachers. For instance, Indonesia teachers had difficulties justifying why the procedure of fraction division needs to 'flip and multiply' (Julie, 2017) or American teachers still had difficulties in knowing the referent units and partitioning (Izsák et al., 2019).

Two recent studies with bigger samples of teachers across the U.S. indicate that the conceptual understanding of fraction division has remained challenging for American teachers (Copur-Gencturk, 2021, 2022). In 2021, she studied a sample of 303 teachers of the 4^{th} and 5^{th} grades from across the U.S. to examine their understanding of how fraction addition and division algorithms work. The sample was recruited from across the U.S. via email. Only 11.0% of the teachers provided an explanation for how the division algorithm worked and how to make sense of the quotient. Teachers holding credentials in other subjects (i.e., special education) performed worse than those holding multiple subject teaching credentials (i.e., generalist teachers). In 2022, Copur-Gencturk continued to study 603 U.S. elementary and middle school teachers who were teaching fractions in grades 3 to 7. The teachers were given two estimation tasks by answering where the answer would fall in the range of the number line without calculations. One question was fraction addition, and the other was fraction division. The results indicated that the teachers' estimations were only partially accurate and reasonable, particularly when fraction division was involved. The reason for the poorer performance on fraction division estimation resonates with the concept of 'the whole number bias' (Ni, 2005) which is the overgeneralization of the rules in the division of whole numbers to fractions. Again, when compared with teachers with different credentials, SETs performed the lowest in fraction estimation tasks.

Given the fact that the research is so limited to fractions of SETs. The two recent studies with the large USA in-service teacher samples above point out the fact that the knowledge of fractions and fraction division is the lowest among other groups of teachers. These findings are alarming. In addition, the picture of the PTs in special education is not rosy either. Morano and Riccomini (2020) reported that no PTs of general or special education could come up with an accurate word problem for the fraction. Two general education PTs and no special education PT could use visual representations to explain the fraction division problems. More special than general education PTs did not attempt or did not complete a division story problem (71% vs. 52%). In sum, the poor performance of fraction division and the learning or learned helplessness (blank answers) from the SETs and PTs in special education suggest an urgent need for quality PD in mathematics education that can cater to the needs of teachers in special education.

Conceptual Understanding of Fraction Division via Problem Posing

Problem posing (PP) in mathematics education means writing or creating story problems. Research shows that PP has many benefits in learning mathematics as follows: a. supports learners' meaningful learning, creates opportunities to avoid conceptual errors and enhance conceptual understanding; b. enhances mathematical creative thinking; c. serves as a diagnostic tool to study the learner's conceptual understanding, misconceptions, or difficulties, particularly through error analyses; d. serves as a remediation, professional development or research tool to study teachers' learning, knowledge, or beliefs (Leung & Carbone, 2013; Ma, 2010; Osana & Pelczer, 2015; Xie & Masingila, 2017; Yao et al., 2021). There are many research methods studying the topic of fraction divisions ranging from multiple choices of scenarios for a fraction problem, to solving word problems, PP, and a recent trend of testing the understanding of fraction magnitudes on a number line. Ball (1990) tested a group of PTs on the question of 1¾÷½. A multiple-choice test revealed that around one-third of the teachers answered correctly (31.3% and 40% of elementary and secondary teachers, respectively). However, the follow-up justifications offered by the teachers based on their stories showed that no elementary teachers and four secondary teachers could correctly justify their answers. A study by Wu and An (2008) show that 97% of pre-service teachers created word problems for fraction division illogically, with misconceptions, in the pre-test. After learning Model-Strategy-Application (MSA) approach that engaged students in creating various visual and meaningful mathematics models to convey abstract math ideas thru conceptual understanding; developing various strategies to achieve fluency in computations, and developing strategic competence in creating and solving word problems in applications, only 26% of students in the post-test still had difficulties in creating a correct and logical word problem in fraction division. This learning process allow students "to explore the word problems in a manner that encourages more in depth understanding of the word problem while applying mathematics to real life applications" (Wu & An, 2016, p.55).

Therefore, asking the teachers to offer a story about the fraction division problem is an ideal and more thorough way to examine the conceptual understanding of fraction division.

The Strategies and Misconceptions for the Concepts of Fraction Division

There are several models for comprehending fraction division summarized as follows and used for the analyses of this study (Greer, 1992, cited in Olanoff et al., 2014; Ma, 2010). They are 1) partition (the given quantity is equally divided into a given number of groups, or equal sharing; e.g., 'If 1¾ is half of the unit, what is the whole?'). 2) measurement (how many groups of the intended quantity are contained in the given quantity, or how many portions can be made; e.g., 'How many servings of ½ pizza can be made from 1¾ pizzas?). 3) product/area (rectangular model) and factors, e.g., 'If one side of 1¾ square meter rectangle is ½ meter, what is the length of the other side?). Research indicates that referent units, multiple meanings of fraction division as the models shown above, connections between the fraction division with the real-world contexts, and other concepts of operations involved in fraction division are the common difficulties causing misconceptions in understanding fraction divisions (e.g., Leung & Carbone, 2013; Lo & Lou, 2012; Ma, 2010; Olanoff et al., 2014; Stohlmann et al., 2019).

The partitive model is commonly used for the whole number division which is understood as equal sharing. For example, 'we have 24 apples, if shared by 6 people, how many can one get? Or we have 24 students, formed into 6 groups, how many people are in each group? In this case that a partitive model is 'finding the value of a unit when the value of several units is known' (Ma, 2010, p. 75,). However, in fraction division, the condition has reversed. The value of a part of the unit is given, but the whole is the answer to get. For instance, 'If 1 and ¾ dollars can buy ½ of the cake, how much is the full cake?' Thus, the partitive model of fractions is to find the whole when a part of it is known. Thus, when one thinks of fraction division using the partitive model of the whole number to interpret the problem, the story produced would be senseless in the real world, for example, to have a fraction of people.

In addition, the language for fractions and particularly fraction division is important. However, it is usually found that teachers lack the appropriate language to use when solving fraction division problems, or connecting with the fraction problems (Ni, 2001; Tobias, 2013). As a result, the stories created by the teachers when asked to provide a real story to justify the fraction division equation of 'dividing by ½' is often confused with 'dividing by 2' or 'multiplying by ½ or 2' (Ball, 1990; Ma, 2010; McAllister & Beaver, 2012; Nillas, 2010). In addition, PTs tend to see fractions as a part-of-a-whole relationship instead of seeing a fraction as a quantity of which understanding is limited to having whole numbers as the divisor (Lo & Luo, 2012; Xie & Masingila, 2017).

In summary, considering that in-service SETs are understudied in mathematics education, specifically in the context of fraction division, a topic known to trouble many teachers and students as research has suggested. It is crucial for us to gain a comprehensive understanding of this topic through this paper.

Method

Participants

This study involved 74 in-service teachers enrolling in a 3-year master's program in special education at a university in Taiwan in the summer of 2015. The classes were master's programs from years one, two, and three together. Six male and 68 female teachers volunteered for this study coming from different cities s in Taiwan and were categorized into 4 regions. Twenty-eight were from northern Taiwan, eight from the central, 25 from the southern, and 13 from the eastern part of Taiwan. The average teaching year was M = 5.3 (*SD* =3.9) years, ranging from 1 to 22 years. The types of classes in which these teachers were teaching included 4 regular education classes, 24 special classes, 31 resource rooms, and 15 tour teachers.

Tasks

The research tool used in this study was a written test consisting of 10 mathematical problems mixed with rational number arithmetic and word problems. The teachers were asked to solve a written test lasting about an hour. If the teachers needed extra time, they were allowed to do so. This paper only analyzed one question from the test battery. We specifically chose the question, $1\frac{3}{4} \div \frac{1}{2}$ to investigate the mathematical understanding of SETs because this equation is more conceptually and computationally demanding. Moreover, Ma (2010) also used it to study in-service teachers, suggesting that it would be a reliable basis for our study of in-service SETs. The analysis included two parts. The first was the procedural knowledge related to solving the fraction division equation. The second was the conceptual knowledge based on posing a word problem for the fraction division equation solved.

Data Coding

The procedural knowledge was scored based on three levels: 1) complete procedure with the correct answer (in simplest form, proper fraction, 2 points); 2) incomplete procedure with the correct answer (but not in proper fraction form, 1 point), and 3) wrong answer/blanks (0 points). The conceptual knowledge was analyzed by the models used, topics adopted, and misconceptions involved in the problem-posing of the fraction division equation. The data coding involved two graduate students and the main researcher. The value of Cohen's kappa for the two coders on the problem-posing task was 0.93.

Results

First, the analyses of the procedural knowledge (see Table 1) showed that 88% of the teachers could calculate the problem correctly, but only 30% of them did it to the simplest fraction form, i.e., proper fraction, while 58% of teachers wrote down the improper fraction. Usually, the improper fraction as the final answer would be scored as incorrect by school teachers in Taiwan. In addition, the score of 0 can be broken down into two categories. Seven teachers left the answers blank (9%), and three of them wrote a word problem for this question but forgot to answer the arithmetic problem. Two teachers (3%) mistakenly copied the division sign as the multiplication sign after 'flip and multiply' procedure, so reached incorrect answer.

Second, the analyses of conceptual knowledge (see Table 2) showed that 27% of the teachers correctly used the measurement model (see Figures 1 & 2), and 6.8% correctly used the partitive model (see Figure 3) to represent the word problem.

Table 1
Special Education Teachers' Procedural Knowledge of Fraction Division

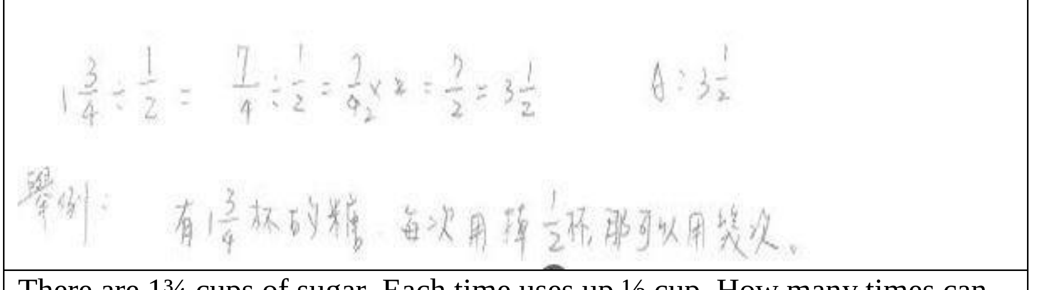
Score	Category	N	%
0	Blank	7	9
0	Wrong answer	2	3
1	Correct answer (non-simplified form)	43	58
2	Correct answer (simplified form)	22	30
Total		74	100.0

Table 2
Special Education Teachers' Conceptual Knowledge of Fraction Division

Correct problems	N	%
measurement	20	27
partitive	5	6.8
sum	25	33.8

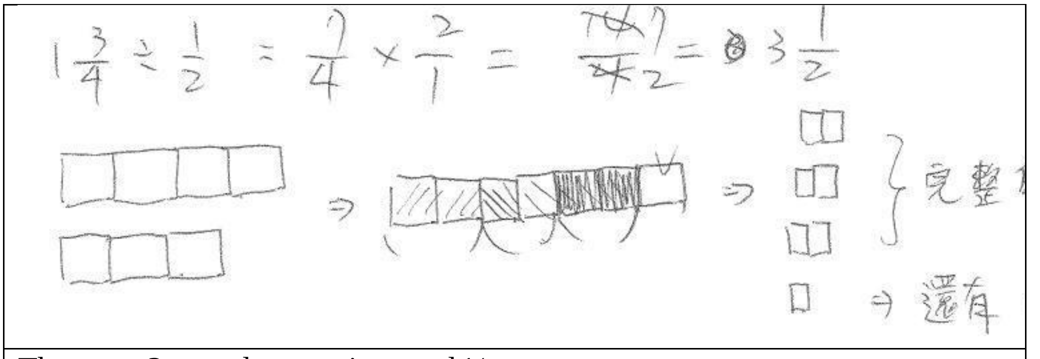
Incorrect problems		
÷ 2	26	35.1
× 2	4	5.4
measurement-incomplete	4	5.4
incomplete story	3	4.1
blank	12	16.2
sum	49	66.2

Figure 1
Sample for the Correct Measurement Model



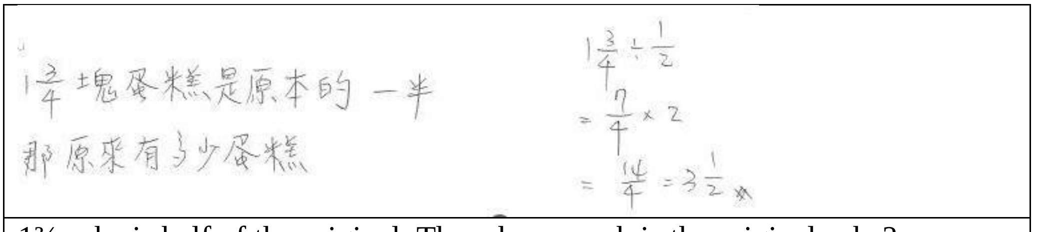
There are 1¾ cups of sugar. Each time uses up ½ cup. How many times can it be used?

Figure 2
Sample for the Correct Measurement Model



There are 3 complete portions and ½.

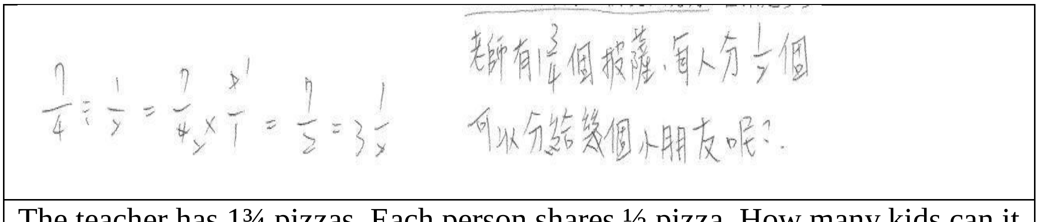
Figure 3
Sample for the Correct Partitive Model



1¾ cake is half of the original. Then, how much is the original cake?

One teacher came up with an example using a measurement model, but the answer was senseless in a real-life context to have a fraction number of kids (3½). Her example was ‘The teacher has 1¾ pizzas. Each person shares ½ pizza. How many kids can it be shared with?’ (Figure 4). To sum up, a total of 33.8% of the teachers demonstrated a correct conceptual understanding of fraction division (see Table 2).

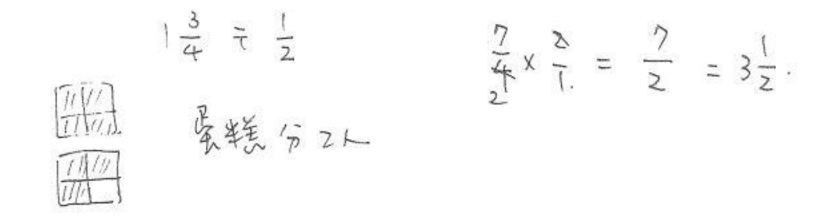
Figure 4
Sample for the Incorrect Measurement Model



The teacher has 1¾ pizzas. Each person shares ½ pizza. How many kids can it be shared with?

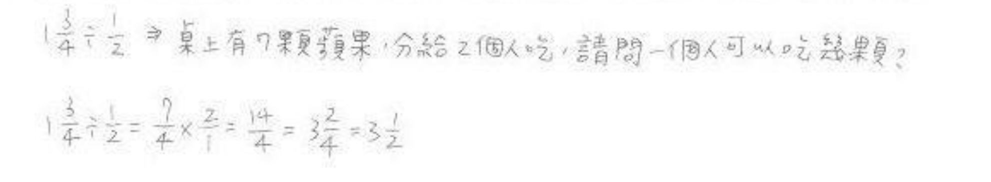
On the other hand, 66.2% of teachers failed to pose a correct word problem. With further analyses, the results showed that 35.1% of stories were confused with 'division by 2' (Figures 5 & 6) and 5.4% with 'multiplication by 2' (Figure 7) for stories.

Figure 5
Sample for the Incorrect Answer – Divided by 2



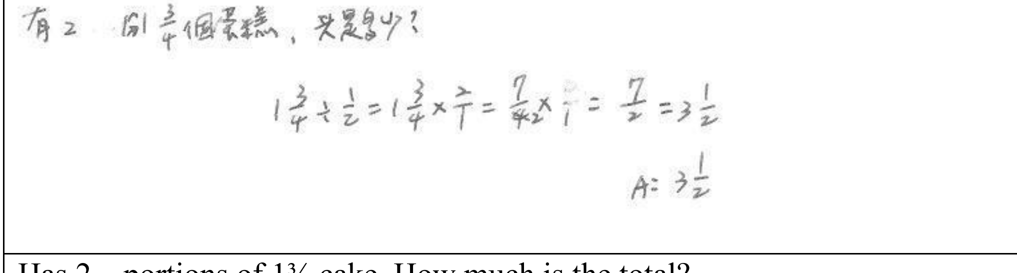
The cake is shared by 2 people.

Figure 6
Sample for the Incorrect Answer – Divided by 2



On the table, there are 7 apples. Share with 2 people. How many apples can each person eat?

Figure 7
Sample for the Incorrect Answer – Multiply by 2



Has 2…portions of 1¾ cake. How much is the total?

Twelve (16.2%) teachers left the question blank (Figure 8), whereas seven (9.5%) teachers tried to answer the question but with incomplete stories (Figure 9). Four teachers (5.4%) tried to use the measurement model to explain, yet they failed to complete the story. If adding these four teachers, a total of 32.4% of teachers preferred the measurement models. Furthermore, four teachers (5.4%) failed to answer both parts of the problem and left the question blank (see Figure 10).

Figure 8
Sample for Blank Word Problem – with Correct Calculation

$1\frac{3}{4} \times 2 = \frac{7}{4} \times 2 = \frac{7}{2}$ 答: $\frac{7}{2}$ 意义、例子不知.
The correct calculation, but no idea about the meaning and the example.

Figure 9
Sample for Incomplete Word Problem – Blank Calculation

有1又$\frac{3}{4}$塊蛋糕,
There are 1 and ¾ pieces of cake…

Figure 10
Sample for Double Blank of Calculation and Word Problem

$1\frac{3}{4} \div \frac{1}{2} =$
Copy the problem only without knowing what to do next-solving it and posing a word problem.

Third, the topics adopted by the teachers were summarized in Table 3. Food (40.54%) such as pizza, green onion pancakes, cake, and cookies, was the most favorite topic used as an example for the problem-posing. The other three categories shared the same percentage (12.16%) for cooking ingredients, drinks, and objects, respectively. Examples of drinks could be juice, milk, and water. Cooking ingredients were like flour and sugar for baking. String, ribbon, road, and field/area are categorized as objects. If combining cooking ingredients, drinks, and food into a bigger category as edible material, 64.86% of the special education teachers preferred edible and concrete examples.

Table 3
Topics Used for Fraction Division Word Problems

	MA1	MA2	MA3	Frequency	%
No/incomplete story	3	8	6	17	22.97
food	13	9	8	30	40.54
Cooking ingredients	6	1	2	9	12.16
Drinks	1	3	5	9	12.16
Objects	1	5	3	9	12.16
Total	24	26	24	74	100.00

MA1=Master's year 1, MA2=Master's year 2, MA3=Master's year 3

Discussion and Conclusion

The results of knowledge of fraction division of Taiwanese SETs resonate with the findings on fraction division that teachers' procedural knowledge is stronger than their conceptual knowledge of fraction divisions (Julie, 2017; Ma, 2010; Lo & Lou, 2012; Xie & Masingila, 2017). Although this study is not intended for cross-national comparison, the data from other studies can help us understand the level of knowledge of Taiwanese SETs. The results show that Taiwanese SETs (88%) demonstrated adequate procedural knowledge with correct calculation skills. Chinese teachers (Ma, 2010) were 100% correct for the same problem. Still, 9% of Taiwanese SETs left the question blank. Their helplessness deserves our attention in mathematics education. A follow-up interview that this study lacked and offering PD programs focusing more on what teachers can do and studying their learning trajectory of fraction division is needed (Olanoff et al., 2014)

Regarding conceptual knowledge, Taiwanese SETs in this study performed better than American teachers in Ma's study (2010) (33.8 % vs. 0%). However, 90% of Chinese teachers could generate a correct word problem for the equation. Sixty of 72 (83.3%) Chinese teachers created more than 80 stories representing the meaning of this equation. Six (8%) Chinese teachers had blank answers, and one teacher provided an incorrect story, whereas 12 (16.2%) Taiwanese SETs left the question blank and seven (9.5%) had incomplete stories. When compared with other Taiwanese teachers, the conceptual knowledge of fraction division of Taiwanese SETs was weaker than that of PTs in general education (33.8% vs. 65%) to pose an adequate word problem (Lo & Lou, 2012).

In addition, when we look at the models for the fraction division problem-posing posted by the Chinese teachers' in Ma's study, their representations of word problems look more diversified. Twelve (16.67%) Chinese teachers posed more than one story to explain the multiple meanings of the fraction division equation. This study did not prompt Taiwanese SETs to do so. Thus, it's unclear how many stories could have been generated by each SET. Future studies can investigate this. Chinese teachers used 3 models (measurement, partitive, and factor/product), and Taiwanese SETs used only two, who favored the measurement model (27%, 32.4% if added the incomplete measurement model) to the partitive model (6.8%). Such preference is the same in the findings of Taiwanese PTs (Lo & Lou, 2012). On the contrary, the measurement model was less preferable than the partitive model (20% and 77%) by Chinese teachers (Ma, 2010). Only 4% of them and no Taiwanese SETs used the factor/product model. As Lo and Lou (2012) suggest that the differences could be due to mathematics textbook designs and the learning experiences of the teachers. It is worth comparing the textbooks or teaching practices in the mathematics classroom in the future.

The error analysis of the word problems can pinpoint the difficulties and misconceptions underlining the conceptual understanding of fraction division. Confusions about 'dividing by ½' with 'dividing by 2' seem to be the most common mistake made by SETs in this study. 35.1% of Taiwanese SETs made this error (Table 2). The examples can be seen in Figures 5 and 6. This finding resonates with the data in other studies (Ball, 1990; Copur-Gencturk, 2022. Isik, & Kar, 2012; Ma, 2010; Xie & Masingila, 2017). Ten of 23 (43%) American teachers made the same error (Ma, 2010). The solution procedure of 'times 2' after 'flip and multiply' did not match the semantic meaning of the language used as 'divided by 2'. This could be attributed to the incomplete understanding of fraction divisions by SETs. They tend to associate the meanings of whole number division with fair sharing. It is likely that when they read ½, the meaning and the language of 'half' would channel them to 'divided by 2'. Furthermore, wanting 'half of it' (×½) can be interpreted in the equation as *(*×½ = ÷2). As a result, such an error would occur if SETs do not have a 'profound understanding of fundamental mathematics' (PUFM) as Ma (2010) has emphasized.

In addition, 'multiply by 2' was another error type made by Taiwanese SETs (Figure7). Four (5.4%) Taiwanese SETs made this error, which did not show in Ma's (2010) Chinese or American teachers. Yet, such error was seen in another study of American in-service teachers (Copur-Gencturk, 2022). It is likely that the Taiwanese SETs posed the problem based on the calculation procedures without understanding the meaning of this fraction division equation. As a result, when ÷½ was changed into × 2/1, visually the number 2 allured these teachers to this mistake. Although only four teachers made this error, their misconceptions reflected their lack of conceptual understanding of fraction division should not be ignored.

On the other hand, Ma (2010) indicated that six (26%) American teachers confounded 'multiply by ½' with 'divided by ½', thinking of the word problem as 'take half of the total'. Ma explained that this reflected the incomplete knowledge of fraction multiplication and thus confused it with fraction division.

Yet, this finding was not seen in Taiwanese SETs. This error is found in other studies (Redmond 2009; Tirosh, 2000).

Research has concluded that understanding the fraction division concepts and posing a relevant word problem rely on knowing the part-whole relations, referent units, and referent whole, and the context of a fractional quantity (Morano & Riccomini, 2020; Olanoff et al., 2014). The PD programs should connect all other mathematical concepts relating to fraction division as a 'package' or 'concept knot' (Ma, 2010). For example, the concepts of addition and multiplication of whole numbers, fractions with units, division with whole numbers, multiplication of fractions, and then division with fractions should be integrated and learned as a connected package. In this way, teachers' conceptual knowledge of fraction division will be consolidated, and mistakes will be decreased greatly.

In fact, mathematical understanding requires precise language. Tobias (2013) found that many PTs had difficulty distinguishing among the terms " of, " "of one," "of the," and " of each" which impeded their understanding of fraction division. Problem posing requires using precise language to express the mathematical concepts behind the equation. This process demands the knowledge of prepositional phrases, conjunctions, and modifying clauses (Tobias, 2013; McAllister & Beaver, 2012). Yao et al. (2021) found that adding a language clue could enhance PTs' conceptual understanding than graphical representations in problem-posing. Hohensee and Jansen (2017) suggest that 'equal sharing' be changed to 'distribution' for the division can enhance the understanding of the measurement model for fraction division and minimize mistakes. Teachers need to practice creating well-written and meaningful fraction word problems to offer their students a rich and authentic learning environment. Further research on mathematical language adopting problem-posing strategies in PD programs would be a promising endeavor (Yao et al., 2021)

Like the American teachers (Ma, 2010), Taiwanese SETs also preferred edible materials, such as food and drinks which are concrete and visible as examples. In contrast, Chinese teachers' examples were more diversified, abstract, and invisible. The examples ranged from the length of the road, and the area of land to the time to finish work and travel time. Such competence requires a PUFM from the teachers to connect various examples in life and make students understand that the same equation of fraction division can be applied to various contexts in life. Special needs students shall not be limited to concrete and simple examples from the teachers. Ma (2010) warned us that "without a solid knowledge of what to represent, no matter how much one is motivated to connect mathematics with students' lives, one still cannot produce a conceptually correct representation" (p. 82). In short, motivation and fun cannot replace PUFM.

Faulkner and Cain (2013) ,the only current study on the professional development (PD) of SETs on fractions, point out that SETs seldom have PD in mathematics as regular education teachers do. This finding might lead to the results in Copur-Gencturk's big-scale studies across the US (2021, 2022) that SETs had more difficulties than regular education teachers in fraction division. In addition, mathematical remediation in special education tends to overemphasize calculation accuracy/fluency over conceptual understanding, and fraction division only recently gained more attention in the research of special education (Jordan et al., 2019). Therefore, SETs are very likely not receiving enough PD focusing on conceptual development as PTs or at work. Jordan et al.'s (2016) longitudinal study discovered that conceptual understanding rather than calculation fluency or multiplication skills can predict better failure in the 6^{th} grade for children with mathematics difficulties. More PD work with SETs focusing on conceptual understanding of fractions has great potential to narrow the achievement gap between the special education students and their counterparts in regular education.

In conclusion, studying the mathematical knowledge of SETs should be encouraged. Fraction division remains a challenging topic for both students and teachers in school mathematics. Failure to master it could impede students' learning of algebra and continue higher education, limit employment, and live fully in a world with the increasing demand for numeracy interpretations. The future PD should focus more on concepts than procedures for SETs. In addition, research should focus more on what teachers can do, and the processes of how teachers overcome difficulties and develop a conceptual understanding (Olanoff et al, 2014). Only when SETs reach the state of PUFM will the special education (students/teachers) be truly on the balanced scale of (mathematics) education equity.

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Author Notes
Correspondence concerning this article should be addressed to Hsiu-fei Lee, Department of Special Education, National Taitung University, TC215, 369, University Rd. Sec. 2, Taitung 950, Taiwan. Email: fei@nttu.edu.tw

Authors:

Hsiu-fei Lee
Department of Special Education, National Taitung University
Email: fei@nttu.edu.tw

Shuhua An
Department of Teacher Education , California State University, Long Beach
Email: shuhua.an@csulb.edu

Jane Lo
Department of Mathematics, Western Michigan University
Email: jane-jane.lo@wmich.edu